جامـــــعـة الـمـــــسـتـقـبـل
AL MUSTAQBAL UNIVERSITY

كلية العلـوم
قســـم الادلة الجنائية

المحاضرة السادسة والسابعة

Method of Integration

المادة : حساب التفاضل والتكامل
المرحلة : الاولى
اسم الاستاذ: م.م ريام ثائر احمد

1- Integration by parts:

The formula for integration by parts comes from the product rule:-

$$d(u\cdot v)=u\cdot dv+v\cdot du \quad\Rightarrow\quad u\cdot dv=d(u\cdot v)-v\cdot du$$

and integrated to give: $\int u\,dv=\int d(u\cdot v)-\int v\,du$

then the integration by parts formula is:-

$$\int u\,dv=u\cdot v-\int v\,du$$

Rule for choosing u and dv is:

For u: choose something that becomes simpler when differentiated.

For dv: choose something whose integral is simple.

It is not always possible to follow this rule, but when we can.

EX-1 – Evaluate the following integrals:

1) $\int xe^{x}\,dx$

2) $\int x\cdot \cos x\,dx$

3) $\int \frac{x}{\sqrt{x-1}}\,dx$

4) $\int x^{2}\cdot \ln x\,dx$

5) $\int x\cdot \sec^{2}x\,dx$

6) $\int \ln\left(x+\sqrt{1+x^{2}}\right)dx$

7) $\int \sin^{-1}ax\,dx$

8) $\int e^{ax}\cdot \sin bx\,dx$

9) $\int x^{3}\cdot e^{x}\,dx$

10) $\int x^{3}\cdot e^{x^{2}}\,dx$

Sol. –

1) let $\left.\begin{aligned} u=x &\Rightarrow du=dx \\ dv=e^{x}dx &\Rightarrow v=e^{x}\end{aligned}\right\}\Rightarrow \int u\,dv=u\cdot v-\int v\,du$

$$\int x\cdot e^{x}dx=x\cdot e^{x}-\int e^{x}dx=x\cdot e^{x}-e^{x}+c$$

2) let $\left.\begin{array}{l} u = x \Rightarrow du = dx \\ dv = \cos x\,dx \Rightarrow v = \sin x \end{array}\right\} \Rightarrow \int u\,dv = u \cdot v - \int v\,du$

$$\int x \cdot \cos x\,dx = x \cdot \sin x - \int \sin x\,dx = x \cdot \sin x + \cos x + c$$

3) let $\left.\begin{array}{l} u = x \Rightarrow du = dx \\ dv = \dfrac{1}{\sqrt{x-1}}dx \Rightarrow v = 2(x-1)^{1/2} \end{array}\right\} \Rightarrow \int u\,dv = u \cdot v - \int v\,du$

$$\int \frac{x}{\sqrt{x-1}}dx = 2x \cdot (x-1)^{1/2} - 2\int (x-1)^{1/2}dx$$

$$= 2x \cdot \sqrt{x-1} - \frac{2(x-1)^{3/2}}{3/2} + c = 2x \cdot \sqrt{x-1} - \frac{4}{3}\sqrt{(x-1)^3} + c$$

4) let $\left.\begin{array}{l} u = \ln x \Rightarrow du = \dfrac{1}{x}dx \\ dv = x^2 dx \Rightarrow v = \dfrac{x^3}{3} \end{array}\right\} \Rightarrow \int u\,dv = u \cdot v - \int v\,du$

$$\int x^2 \cdot \ln x\,dx = \frac{x^3}{3} \cdot \ln x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3} \cdot \ln x - \frac{1}{9}x^3 + c$$

5) let $\left.\begin{array}{l} u = x \Rightarrow du = dx \\ dv = \sec^2 x\,dx \Rightarrow v = \tan x \end{array}\right\} \Rightarrow \int u\,dv = u \cdot v - \int v\,du$

$$\int x \cdot \sec^2 x\,dx = x \cdot \tan x - \int \tan x\,dx = x \cdot \tan x + \ln|\cos x| + c$$

6) let $u = \ln\left(x + \sqrt{1+x^2}\right) \Rightarrow du = \dfrac{1 + \dfrac{2x}{2\sqrt{1+x^2}}}{x + \sqrt{1+x^2}}dx$

$dv = dx \Rightarrow v = x$

$$\int \ln\left(x + \sqrt{1+x^2}\right)dx = x \cdot \ln\left(x + \sqrt{1+x^2}\right) - \int x(1+x^2)^{-1/2}dx$$

$$7)\quad let\ \ u=\sin^{-1}ax \Rightarrow du=\frac{a\,dx}{\sqrt{1-a^2x^2}} \quad \& \quad dv=dx \Rightarrow v=x$$

$$\int \sin^{-1}ax\,dx = x\cdot\sin^{-1}ax-\int\frac{a\,x}{\sqrt{1-a^2x^2}}dx$$

$$=x\cdot\sin^{-1}ax+\frac{1}{2a}\int -2a^2x\left(1-a^2x^2\right)^{-1/2}dx$$

$$=x\cdot\sin^{-1}ax+\frac{1}{2a}\cdot\frac{\left(1-a^2x^2\right)^{1/2}}{1/2}+c=x\cdot\sin^{-1}ax+\frac{\sqrt{1-a^2x^2}}{a}+c$$

$$8)\quad let\ \ u=e^{ax} \Rightarrow du=a\cdot e^{ax}\,dx \quad \& \quad dv=\sin bx\,dx \Rightarrow v=-\frac{1}{b}\cos bx$$

$$\int e^{ax}\cdot\sin bx\,dx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b}\int e^{ax}\cdot\cos bx\,dx \quad ....(1)$$

$$let\ \ u=e^{ax} \Rightarrow du=a\cdot e^{ax}\,dx \quad \& \quad dv=\cos bx\,dx \Rightarrow v=\frac{1}{b}\sin bx$$

$$\int e^{ax}\cdot\cos bx\,dx=\frac{1}{b}e^{ax}\cdot\sin bx-\frac{a}{b}\int e^{ax}\cdot\sin bx\,dx \quad ....(2)$$

sub. (2) in (1) $\Rightarrow$

$$\int e^{ax}\cdot\sin bx\,dx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx\,dx-\frac{a^2}{b^2}\int e^{ax}\cdot\sin bx\,dx$$

$$\int e^{ax}\cdot\sin bx\,dx+\frac{a^2}{b^2}\int e^{ax}\cdot\sin bx\,dx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx\,dx+c$$

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cdot\sin bx\,dx=\frac{e^{ax}}{b^2}\left(a\sin bx-b\cos bx\right)+c$$

$$\therefore\quad \int e^{ax}\cdot\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(a\sin bx-b\cos bx\right)+c$$

9) <u>derivative of u</u>   <u>integration of dv</u>

derivative of u		integration of dv
x^3	+	e^x
$3x^2$	-	e^x
$6x$	+	e^x
6	-	e^x
0		e^x

$$\therefore\ \int x^3e^{ax}\,dx=x^3e^x-3x^2e^x+6xe^x-6e^x+c$$

$$=e^x\left(x^3-3x^2+6x-6\right)+c$$

$10)$ let $u = x^2 \Rightarrow du = 2x\,dx$ & $dv = x \cdot e^{x^2} dx \Rightarrow v = \frac{1}{2} e^{x^2}$

$$\int x^3 \cdot e^{x^2} dx = \frac{1}{2} x^2 \cdot e^{x^2} - \frac{1}{2} \int 2x \cdot e^{x^2} dx = \frac{1}{2} x^2 \cdot e^{x^2} - \frac{1}{2} e^{x^2} + c$$

2- <u>Odd and even powers of sine and cosine</u>:

To integrate an odd positive power of $sinx$ (say $sin^{2n+1}x$) we split off a factor of $sinx$ and rewrite the remaining even power in terms of the cosine. We write:-

$$\int sin^{2n+1} x \cdot dx = \int \left(1 - cos^2 x\right)^n \cdot sin\,x \; dx$$

and $$\int cos^{2n+1} x \cdot dx = \int \left(1 - sin^2 x\right)^n \cdot cos\,x \; dx$$

***<u>EX-2</u>*- Evaluate:**

1) $\int sin^3 x\, dx$ 2) $\int cos^5 x\, dx$

***<u>Sol.</u>*-**

$$1)\; \int sin^3 x\, dx = \int sin^2 x \cdot sinx\, dx = \int \left(1 - cos^2 x\right) \cdot sinx\, dx$$
$$= \int sinx\, dx + \int cos^2 x \cdot (-sinx)\, dx = -cos\,x + \frac{1}{3} cos^3 x + c$$

$$2)\; \int cos^5 x\, dx = \int cos^4 x \cdot cosx\; dx = \int \left(1 - sin^2 x\right)^2 \cdot cos\,x\, dx$$
$$= \int cosx\, dx - 2\int sin^2 x \cdot cos\,x\, dx + \int sin^4 x \cdot cos\,x\, dx$$
$$= sinx - \frac{2}{3} sin^3 x + \frac{1}{5} sin^5 x + c$$

To integrate an even positive power of sine (say $sin^{2n}x$) we use the relations:-

$$cos^2\theta = \frac{1 + cos\,2\theta}{2} \qquad or \qquad sin^2\theta = \frac{1 - cos\,2\theta}{2}$$

then we can write:-

$$\int \sin^{2n} x \cdot dx = \int \left(\frac{1-\cos 2x}{2} \right)^n \, dx$$

$$\text{and } \int \cos^{2n} x \cdot dx = \int \left(\frac{1+\cos 2x}{2} \right)^n \, dx$$

EX-3- Evaluate:

1) $\int \cos^2 \theta \, d\theta$ 2) $\int \sin^4 \theta \, d\theta$

Sol.-

$$1)\ \int \cos^2\theta \, d\theta = \int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}\left[\int d\theta + \frac{1}{2}\int 2\cos 2\theta \, d\theta \right]$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta \right] + c$$

$$2)\ \int \sin^4\theta \, d\theta = \int \left(\frac{1-\cos 2\theta}{2} \right)^2 d\theta = \frac{1}{4}\left[\int d\theta - \int \cos 2\theta (2d\theta) + \int \cos^2 2\theta \, d\theta \right]$$

$$= \frac{1}{4}\left[\theta - \sin 2\theta + \int \frac{1+\cos 4\theta}{2} d\theta \right] = \frac{1}{4}\left[\theta - \sin 2\theta + \frac{1}{2}(\theta + \frac{1}{4}\sin 4\theta) \right] + c$$

$$= \frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + c$$

To integrate the following identities:-

$$\int \sin mx \cdot \sin nx \ dx \ , \int \sin mx \cdot \cos nx \ dx \ , \text{and } \int \cos mx \cdot \cos nx \ dx$$

To integrate the following identities:-

$$\int \sin mx \cdot \sin nx \ dx \ , \int \sin mx \cdot \cos nx \ dx \ , \text{and } \int \cos mx \cdot \cos nx \ dx$$

we use the following formulas:-

$$\sin mx \cdot \sin nx = \frac{\cos(m-n)x - \cos(m+n)x}{2}$$

$$\sin mx \cdot \cos nx = \frac{\sin(m-n)x + \sin(m+n)x}{2}$$

$$\cos mx \cdot \cos nx = \frac{\cos(m-n)x + \cos(m+n)x}{2}$$

EX-4- Evaluate:

$1)\ \int \sin 3x \cdot \cos 5x\, dx \qquad 2)\ \int \cos x \cdot \cos 7x\, dx \qquad 3) \int \sin x \cdot \sin 2x\, dx$

Sol.-

$$1)\ \int \sin 3x \cdot \cos 5x\, dx = \frac{1}{2}\int \left(\sin(3x-5x)+\sin(3x+5x)\right)dx$$

$$= \frac{1}{2}\left[-\frac{1}{2}\int \sin 2x(2dx)+\frac{1}{8}\int \sin 8x(8dx)\right] = \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + c$$

$$2)\ \int \cos x \cdot \cos 7x\, dx = \frac{1}{2}\int \left(\cos(6x)+\cos(8x)\right)dx = \frac{1}{12}\sin 6x + \frac{1}{16}\sin 8x + c$$

$$3)\ \int \sin x \cdot \sin 2x\, dx = \frac{1}{2}\int \left(\cos x - \cos 3x\right)dx = \frac{1}{2}\sin x - \frac{1}{6}\sin 3x + c$$

6-3- Trigonometric substitutions:

Trigonometric substitutions enable us to replace the binomials a^2-u^2 , a^2+u^2 , and u^2-a^2 be single square terms. We can use:-

$$u = a\sin\theta \quad for \quad a^2-u^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$$

$$u = a\tan\theta \quad for \quad a^2+u^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

$$u = a\sec\theta \quad for \quad u^2-a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta-1) = a^2\tan^2\theta$$

EX-5 Evaluate the following integrals:

$1)\ \int \frac{z^5\, dz}{\sqrt{1+z^2}}$

$2)\ \int \frac{dx}{\sqrt{4+x^2}}$

$3)\ \int \frac{dx}{4-x^2}$

$4)\ \int \frac{x^2}{\sqrt{9-x^2}}dx$

$5)\ \int \frac{dt}{\sqrt{25t^2-9}}$

$6)\ \int \frac{dy}{\sqrt{25+9y^2}}$

Sol.-

1) let $z = \tan\theta \quad \Rightarrow \quad dz = \sec^2\theta \cdot d\theta \qquad \tan\theta = \frac{z}{1}$

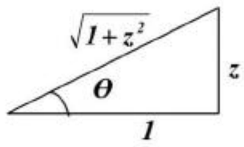
$$\int \frac{z^5\,dz}{\sqrt{1+z^2}} = \int \frac{\tan^5\theta \cdot \sec^2\theta\ d\theta}{\sqrt{1+\tan^2\theta}} = \int \tan^5\theta \cdot \sec\theta\ d\theta$$

$$= \int \tan\theta \cdot \sec\theta\left(\sec^2\theta - 1\right)^2 d\theta$$

$$= \int \sec^4\theta(\tan\theta \cdot \sec\theta\,d\theta) - 2\int \sec^2\theta(\tan\theta \cdot \sec\theta\,d\theta) + \int \tan\theta \cdot \sec\theta\,d\theta$$

$$= \frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta + c$$

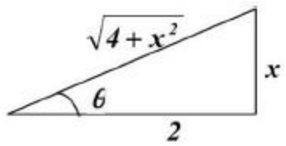
$$= \frac{1}{5}(\sqrt{1+z^2}\,)^5 - \frac{2}{3}(\sqrt{1+z^2}\,)^3 + \sqrt{1+z^2} + c$$



2) let $x = 2\tan\theta \quad \Rightarrow \quad dx = 2\sec^2\theta \cdot d\theta \qquad \tan\theta = \frac{x}{2}$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\ d\theta}{\sqrt{4+4\tan^2\theta}} = \int \sec\theta\ d\theta = \ln\left|\sec\theta + \tan\theta\right| + c$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c$$

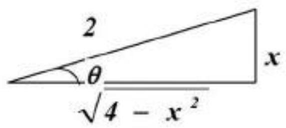


$$= \ln\left|\sqrt{4+x^2} + x\right| + c' \qquad \text{where } c' = c - \ln 2$$

3) let $x = 2\sin\theta \quad \Rightarrow \quad dx = 2\cos\theta \cdot d\theta$

$$\int \frac{dx}{4-x^2} = \int \frac{2\cos\theta\ d\theta}{4-4\sin^2\theta} = \frac{1}{2}\int \frac{d\theta}{\cos\theta} = \frac{1}{2}\int \sec\theta\,d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + c$$



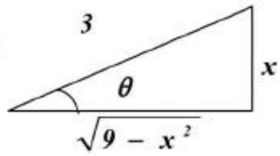
$$= \frac{1}{2}\ln\left|\frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{2+x}{\sqrt{(2-x)(2+x)}}\right| + c = \frac{1}{2}\ln\left|\sqrt{\frac{2+x}{2-x}}\right| + c = \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + c$$

4) let $x = 3\sin\theta \quad \Rightarrow \quad dx = 3\cos\theta \cdot d\theta$

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,d\theta = 9\int \sin^2\theta\,d\theta$$

$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + c$$



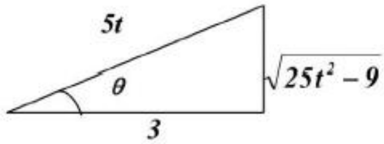
$$= \frac{9}{2}\left(\theta - \sin\theta\cdot\cos\theta\right) + c$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + c = \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\cdot\sqrt{9-x^2} + c$$

5) let $5t = 3\sec\theta \quad \Rightarrow \quad 5dt = 3\sec\theta\cdot\tan\theta\,d\theta$

$$\int \frac{dt}{\sqrt{25t^2-9}} = \int \frac{3/5\,\sec\theta\cdot\tan\theta\,d\theta}{\sqrt{9\sec^2\theta - 9}} = \frac{1}{5}\int \sec\theta\,d\theta$$

$$= \frac{1}{5}\ln\left|\sec\theta + \tan\theta\right| + c$$



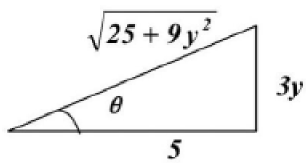
$$= \frac{1}{5}\ln\left|\frac{5t}{3} + \frac{\sqrt{25t^2-9}}{3}\right| + c$$

$$= \frac{1}{5}\ln\left|5t + \sqrt{25t^2-9}\right| + c' \quad \text{where } c' = c - \frac{1}{5}\ln 3$$

6) let $3y = 5\tan\theta \quad \Rightarrow \quad 3dy = 5\sec^2\theta\,d\theta$

$$\int \frac{dy}{\sqrt{25+9y^2}} = \int \frac{5/3\,\sec^2\theta\,d\theta}{\sqrt{25+25\tan^2\theta}} = \frac{1}{3}\int \sec\theta\,d\theta$$

$$= \frac{1}{3}\ln\left|\sec\theta + \tan\theta\right| + c$$



$$= \frac{1}{3}\ln\left|\frac{\sqrt{25+9y^2}}{5} + \frac{3y}{5}\right| + c$$

$$= \frac{1}{3}\ln\left|\sqrt{25+9y^2} + 3y\right| + c' \quad \text{where } c' = c - \frac{1}{3}\ln 5$$