# Integration Methods :- طرق التكامل

## ① Integration By Parts :- التكامل بالاجزاء

This method is used when there is an association of two functions, one of which is not a derivative of the other.

وتستخدم طريقة التكامل بالاجزاء عندما يكون لدينا دالتين نريد تكاملهما وليس لاحداهما مشتقة للاخرى مثل :-

a- $\int x e^{x} dx$

b- $\int t \sin t \, dt$

c- $\int e^{\theta} \cos\theta \, d\theta$

d- $\int x \ln x \, dx$

## Derivative of Integration by parts Formula :- اشتقاق علاقة التكامل بالاجزاء

From the product rule of differentiation

من قانون مشتقة ضرب دالتين

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

where $u$ & $v$ are both functions of $x$

حيث كل من $u$ و $v$ هم دالتين للمتغير $x$ .

Rearranging gives;

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrate both side with respect to (w.r.t) $x$, yields,

$$\int u\frac{dv}{\cancel{dx}}\cancel{dx} = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{\cancel{dx}}\cancel{dx}$$

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$$\int \underbrace{u}_{\text{جزء اشتقاق}}\ \underbrace{dv}_{\text{جزء تكامل}} = uv - \int v\, du$$ ← Integration by parts Formula

* Now, which part to make equal to $u$ & which to make equal to $v$??

- السؤال هو، اي جزء من الدالتين يمكن اعتبارها $u$ والاخرى $v$؟

To answer this enquiry, the choice must be such that the "$u$ part" becomes a constant after successive differentiation, & the "$dv$ part" can be integrated from standard integrals.

i.e

* $x$, $t^2$, or $3\theta$ (algebric terms) $\xrightarrow[\text{to be}]{\text{chosen}}$ "$u$ part"

Note!! (ملاحظة مهمة)

The only exception to this rule is when a "$\ln x$" term is involved; in this case $\ln x$ is chosen as the "$u$ part"

● للاجابة على هذا الاستفسار، الاختيار يجب ان يكون بحيث يصبح الجزء $u$ ثابتاً بعد اشتقاقه بشكل متكرر، و جزء $dv$ يمكن تكامله بطرق التكامل القياسية المعروفة.

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## Examples

① Determine $\int x \cos x \, dx$

Solution

Let $u = x \longrightarrow du = dx$

Let $dv = \cos x \, dx \longrightarrow v = \int \cos x \, dx = \sin x$

$$\int \underline{u}\,\underline{dv} = \underline{u}\,\underline{v} - \int \underline{v}\,\underline{du}$$

$$\therefore \int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x - (-\cos x) + C$$

$$\therefore \int x \cos x \, dx = \boxed{x \sin x + \cos x + c} \quad \underline{Ans}$$

② Determine $\int 3t\, e^{2t} \, dt$

Solution

Let $u = 3t \longrightarrow du = 3 \, dt$

Let $dv = e^{2t} dt \longrightarrow v = \int e^{2t} dt = \frac{1}{2} e^{2t}$

substitute into $\int u \, dv = uv - \int v \, du$

$$\therefore \int 3t\, e^{2t} \, dt = (3t)\left(\tfrac{1}{2} e^{2t}\right) - \int \left(\tfrac{1}{2} e^{2t}\right)(3\, dt)$$

$$= \frac{3}{2} t\, e^{2t} - \frac{3}{2} \int e^{2t} \, dt$$

$$= \frac{3}{2} t\, e^{2t} - \frac{3}{2}\left(\frac{e^{2t}}{2}\right) + C$$

$$\int 3t\, e^{2t} \, dt = \boxed{\frac{3}{2} e^{2t}\left(t - \frac{1}{2}\right) + c} \quad \underline{Ans}$$

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(3). Evaluate $\int_0^{\pi/2} 2\theta \sin\theta \, d\theta$

Solution

Let $u = 2\theta \longrightarrow du = 2\,d\theta$
Let $dv = \sin\theta\, d\theta \longrightarrow v = \int \sin\theta\, d\theta = -\cos\theta$

Substitute into $\int u\,dv = uv - \int v\,du$, yields,

$$\int_0^{\pi/2} 2\theta \sin\theta\, d\theta = (2\theta)(-\cos\theta) - \int(-\cos\theta)(2\,d\theta)$$

$$= -2\theta\cos\theta + 2\int\cos\theta\, d\theta$$

$$= -2\theta\cos\theta + 2\sin\theta\Big]_0^{\pi/2}$$

$$= -\cancel{2}\left(\frac{\pi}{\cancel{2}}\right)\cos\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{\pi}{2}\right) - [0 + 2\sin 0]$$

$$= (0 + 2) - (0 + 0)$$

$$\int_0^{\pi/2} 2\theta \sin\theta\, d\theta = \boxed{2} \qquad \text{Ans}$$

(4) Evaluate $\int_0^1 5x\, e^{4x}\, dx$

Solution

Let $u = 5x \longrightarrow du = 5\,dx$
Let $dv = e^{4x}\, dx \longrightarrow v = \int e^{4x}\, dx = \frac{1}{4}e^{4x}$

Substitute into $\int u\,dv = uv - \int v\,du$, gives,

$$\int_0^1 5x\, e^{4x}\, dx = (5x)\left(\frac{1}{4}e^{4x}\right) - \int\left(\frac{1}{4}e^{4x}\right)(5\,dx)$$

$$= \frac{5}{4}x\,e^{4x} - \frac{5}{4}\int e^{4x}\, dx$$

$$= \frac{5}{4}x\,e^{4x} - \frac{5}{4}\left(\frac{e^{4x}}{4}\right) = \frac{5}{4}e^{4x}\left(x - \frac{1}{4}\right)\Big]_0^1$$

$$= \frac{5}{4}e^{4(1)}\left(1 - \frac{1}{4}\right) - \left[\frac{5}{4}\cancel{e^{0}}\left(0 - \frac{1}{4}\right)\right]$$

$$\int_0^1 5x\,e^{4x}\,dx = \frac{15}{16}e^4 - \left(-\frac{5}{16}\right)$$

$$= 51.186 + 0.313 = \boxed{51.5} \quad \underline{Ans}$$

(5) Evaluate $\int x^2 \sin x \, dx$

Solution

Let $u = x^2 \longrightarrow du = 2x\,dx$

Let $dv = \sin x\,dx \longrightarrow v = \int \sin x\,dx = -\cos x$

substitute into $\int u\,dv = uv - \int v\,du$, gives,

$$\int x^2 \sin x\,dx = (x^2)(-\cos x) - \int(-\cos x)(2x\,dx)$$

$$= -x^2\cos x + 2\int x\cos x\,dx$$

The integral of $\int x\cos x\,dx$ is not a "standard integral" & it can only be determined by using the integration by parts formula again,

هنا نلاحظ بأن حد ال $\int v\,du$ نتج تكامل غير قياسي، ولهذا نحتاج
أن نستخدم التكامل بالأجزاء مرة اخرى وهنا الحد فقط.

From Example (1) $\int x\cos x\,dx = x\sin x + \cos x + C$

$$\therefore \int x^2 \sin x\,dx = -x^2\cos x + 2\left[x\sin x + \cos x\right] + C$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

$$= \boxed{(2 - x^2)\cos x + 2x\sin x + C} \quad \underline{Ans}$$

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⑥ Find $\int x \ln x \, dx$

Solution

Let $u = \ln x \longrightarrow du = \frac{1}{x} dx$

let $dv = x\,dx \longrightarrow v = \frac{x^2}{2}$

substitute into $\int u\,dv = uv - \int v\,du$, yields,

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \left(\frac{x^2}{2}\right) \frac{dx}{x}$$

$$= \frac{x^2}{2} \ln x - \int \frac{x}{2} dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

$$= \boxed{\frac{x^2}{4}(2 \ln x - 1) + c} \quad \text{Ans}$$

⑦ Find $\int \ln x \, dx$

Solution

Let $u = \ln x \longrightarrow du = \frac{1}{x} dx$

Let $dv = dx \longrightarrow v = x$

substitute into $\int u\,dv = uv - \int v\,du$, gives,

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx$$

$$= x \ln x - \int dx = x \ln x - x + c$$

$$= \boxed{x(\ln x - 1) + c} \quad \text{Ans}$$

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