Sinusoidal Steady-State Analysis

In previous lectures, we learned that the forced or steady-state response of circuits to sinusoidal inputs can be obtained by using phasors. We also know that Ohm's and Kirchhoff's laws are applicable to ac circuits. In these lectures, we want to see how nodal analysis, mesh analysis, Thevenin's theorem, Norton's theorem, superposition, and source transformations are applied in analyzing ac circuits. Since these techniques were already introduced for dc circuits, our major effort here will be to illustrate with examples.

Analyzing ac circuits usually requires three steps.

Steps to Analyze AC Circuits:

1. Transform the circuit to the phasor or frequency domain.
2. Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.).
3. Transform the resulting phasor to the time domain

Note: *Frequency domain analysis of an ac circuit via phasors is much easier than analysis of the circuit in the time domain*.

Methods of Circuit Analysis

Circuit analysis techniques will be study in these lectures are nodal analysis, mesh analysis, Thevenin's theorem, Norton's theorem, superposition, and source transformations

1. Nodal Analysis

The basis of nodal analysis is **Kirchhoff's current law**. Since KCL is valid for phasors, we can analyze ac circuits by nodal analysis.

Steps to Determine Node Voltages:

1. Select a node as the reference node. Assign voltages to the remaining nodes. The voltages are referenced with respect to the reference node.

2. Apply KCL to each of the nonreference nodes. Use Ohm's law to express the branch currents in terms of node voltages.

3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

Example 1: Find i_x in the circuit of Fig. 1 using nodal analysis.

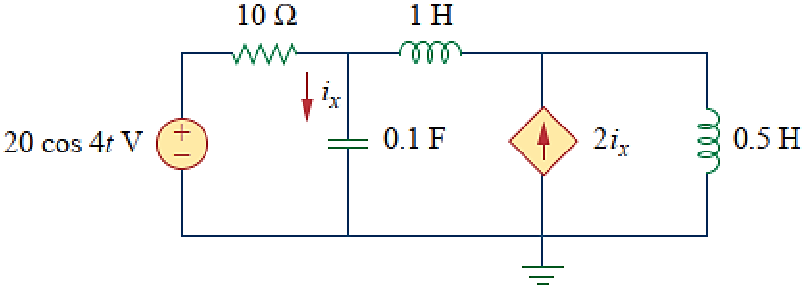


Fig. 1

Solution:

1. convert the circuit to the frequency domain:

$$
\begin{aligned}
20\cos 4t \quad &\Rightarrow \quad 20\angle 0^\circ, \qquad \omega = 4 \text{ rad/s} \\
1\text{ H} \quad &\Rightarrow \quad j\omega L = j4 \\
0.5\text{ H} \quad &\Rightarrow \quad j\omega L = j2 \\
0.1\text{ F} \quad &\Rightarrow \quad \frac{1}{j\omega C} = -j2.5
\end{aligned}
$$

Thus, the frequency domain equivalent circuit is as shown in Fig. 2.

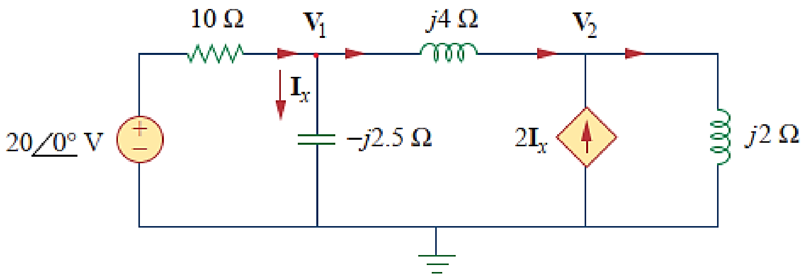


Fig. 2 Frequency domain equivalent of the circuit in Fig. 1

2. Applying KCL at node 1,

$$\frac{20 - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4}$$

or

$$(1 + j1.5)\mathbf{V}_1 + j2.5\mathbf{V}_2 = 20 \qquad \dots (1)$$

At node 2,

$$2\mathbf{I}_x + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

But $\mathbf{I}_x = \mathbf{V}_1/-j2.5$. Substituting this gives

$$\frac{2\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

By simplifying, we get

$$11\mathbf{V}_1 + 15\mathbf{V}_2 = 0 \qquad \dots (2)$$

Equations (1) and (2) can be put in matrix form as:

$$\begin{bmatrix} 1 + j1.5 & j2.5 \\ 11 & 15 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

We obtain the determents as

$$\Delta = \begin{vmatrix} 1 + j1.5 & j2.5 \\ 11 & 15 \end{vmatrix} = 15 - j5$$

$$\Delta_1 = \begin{vmatrix} 20 & j2.5 \\ 0 & 15 \end{vmatrix} = 300, \qquad \Delta_2 = \begin{vmatrix} 1 + j1.5 & 20 \\ 11 & 0 \end{vmatrix} = -220$$

$$\mathbf{V}_1 = \frac{\Delta_1}{\Delta} = \frac{300}{15 - j5} = 18.97\underline{/18.43^\circ}\text{ V}$$

$$\mathbf{V}_2 = \frac{\Delta_2}{\Delta} = \frac{-220}{15 - j5} = 13.91\underline{/198.3^\circ}\text{ V}$$

The current $\mathbf{I}_x$ is given by

$$\mathbf{I}_x = \frac{\mathbf{V}_1}{-j2.5} = \frac{18.97\underline{/18.43^\circ}}{2.5\underline{/-90^\circ}} = 7.59\underline{/108.4^\circ}\text{ A}$$

Transforming this to the time domain,

$$i_x = 7.59 \cos(4t + 108.4^\circ)\text{ A}$$

2. Mesh Analysis

Kirchhoff's voltage law (KVL) forms the basis of mesh analysis. The validity of KVL for ac circuits was shown in the previos lecture and is illustrated in the following examples. Keep in mind that the very nature of using mesh analysis is that it is to be applied to planar circuits.

Steps to Determine Mesh Currents:

1. Assign mesh currents to the n meshes.

2. Apply KVL to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh currents.

3. Solve the resulting n simultaneous equations to get the mesh currents.

Note: *mesh is a loop which does not contain any other loops within it.*

Example 1:

Determine current in the circuit of Fig. 3 using mesh analysis Io.

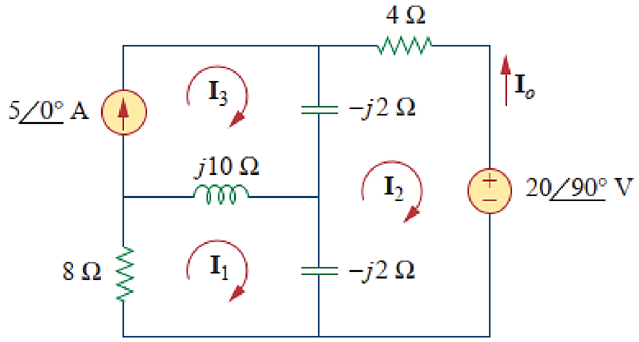


Fig.3

Solution:

Applying KVL to mesh 1, we obtain

$$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3 = 0 \qquad (1)$$

For mesh 2,

$$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\angle 90^\circ = 0 \qquad (2)$$

For mesh 3,
Substituting this in Eqs. (1) and (2), we get

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50 \qquad (3)$$

$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j20 - j10 \qquad (4)$$

Equations (3) and (4) can be put in matrix form as

$$\begin{bmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

from which we obtain the determents

$$\Delta = \begin{vmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{vmatrix} = 32(1 + j)(1 - j) + 4 = 68$$

$$\Delta_2 = \begin{vmatrix} 8 + j8 & j50 \\ j2 & -j30 \end{vmatrix} = 340 - j240 = 416.17\underline{/-35.22^\circ}$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.17\underline{/-35.22^\circ}}{68} = 6.12\underline{/-35.22^\circ}\text{ A}$$

The desired current is

$$\mathbf{I}_o = -\mathbf{I}_2 = 6.12\underline{/144.78^\circ}\text{ A}$$

3. Super Position Theorem

The superposition principle states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the volt- ages across (or currents through) that element due to each independent source acting alone. Since ac circuits are linear, the superposition theorem applies to ac circuits the same way it applies to dc circuits. The theorem becomes important if the circuit has sources operating at different frequencies. In this case, since the impedances depend on frequency, we must have a different frequency domain circuit for each frequency. The total response must be obtained by adding the individual responses in the time domain. It is incorrect to try to add the responses in the phasor or frequency domain. Why? Because the exponential factor is implicit in sinusoidal analysis, and that factor would change for every angular frequency It would therefore not make sense to add responses at different frequencies in the phasor domain. Thus, when a circuit has sources operating at different frequencies, one must add the responses due to the individual frequencies in the time domain.

Steps to Apply Superposition Principle:

1. Turn off all independent sources except one source. Find the output (voltage or current) due to that active source using the techniques covered in points 2 and 3.

2. Repeat step 1 for each of the other independent sources.

3. Find the total contribution by adding algebraically all the contributions due to the independent sources.

Example 1:

Use the superposition theorem to find in the following circuit.

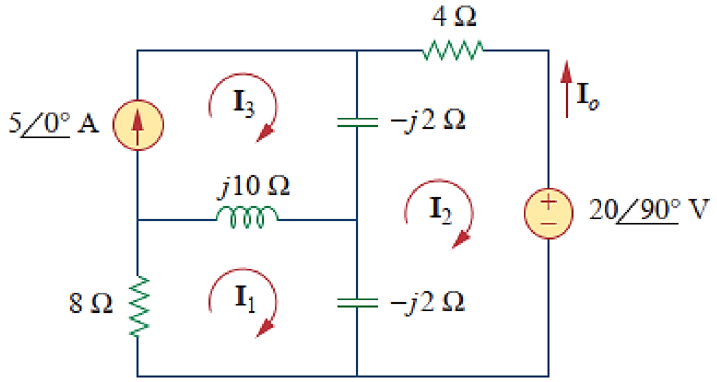


Fig.4

Solution:

Let

$$\mathbf{I}_o = \mathbf{I}'_o + \mathbf{I}''_o \qquad (1)$$

where $\mathbf{I}'_o$ and $\mathbf{I}''_o$ are due to the voltage and current sources, respectively. To find $\mathbf{I}'_o$, consider the circuit in Fig.2(a). If we let Z be the parallel combination of $-j2$ and $8 + j10$, then

$$\mathbf{Z} = \frac{-j2(8 + j10)}{-2j + 8 + j10} = 0.25 - j2.25$$

and current I'_o is

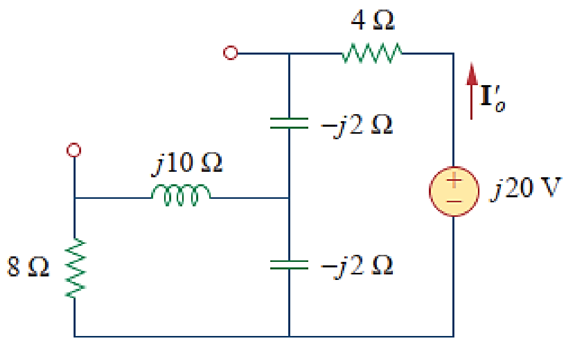
$$\mathbf{I}'_o = \frac{j20}{4 - j2 + \mathbf{Z}} = \frac{j20}{4.25 - j4.25}$$

or

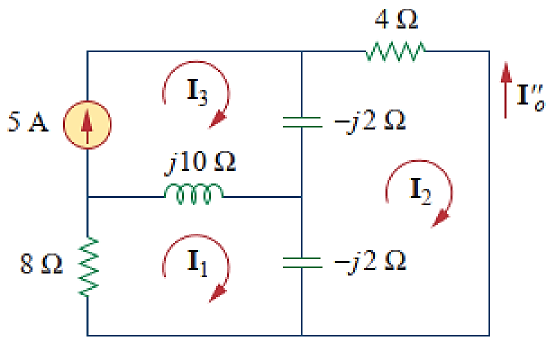
$$\mathbf{I}'_o = -2.353 + j2.353 \qquad (2)$$

To get $\mathbf{I}''_o$ consider the circuit in Fig. 2(b). For mesh 1,

$$(8 + j8)\mathbf{I}_1 - j10\mathbf{I}_3 + j2\mathbf{I}_2 = 0 \qquad (3)$$



(a)



(b)

Fig. 5

For mesh 2,

$$(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j2\mathbf{I}_3 = 0 \qquad (4)$$

For mesh 3,

$$\mathbf{I}_3 = 5 \qquad (5)$$

From Eqs. (4) and (5)

$$(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j10 = 0$$

Expressing $\mathbf{I}_1$ in terms of $\mathbf{I}_2$ gives

$$\mathbf{I}_1 = (2 + j2)\mathbf{I}_2 - 5 \qquad (6)$$

Substituting Eqs. (5) and (6) into Eq. (3), we get

$$(8 + j8)[(2 + j2)\mathbf{I}_2 - 5] - j50 + j2\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_2 = \frac{90 - j40}{34} = 2.647 - j1.176$$

Current $\mathbf{I}_o''$ is obtained as

$$\mathbf{I}_o'' = -\mathbf{I}_2 = -2.647 + j1.176 \qquad (7)$$

From Eqs. (2) and (7), we write

$$\mathbf{I}_o = \mathbf{I}_o' + \mathbf{I}_o'' = -5 + j3.529 = 6.12\underline{/144.78^\circ}\text{ A}$$

4. Source Transformation

A source transformation is the process of replacing a voltage source v_s in series with a resistor R by a current source i_s in parallel with a resistor R, or Vic versa. Source transformation requires the following relationship:

$$\mathbf{V}_s = \mathbf{Z}_s\mathbf{I}_s \quad \Leftrightarrow \quad \mathbf{I}_s = \frac{\mathbf{V}_s}{\mathbf{Z}_s}$$

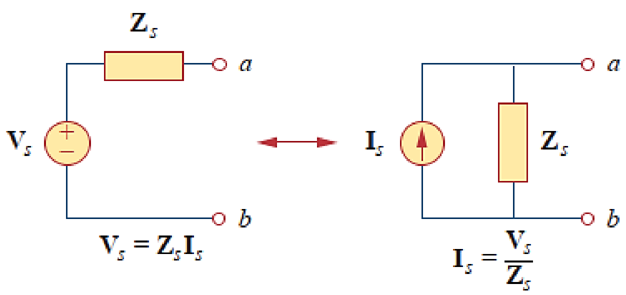


Fig. 6 Source Transformation

Example:

Calculate V_x in the circuit of Fig.7 below using the method of source transformation.

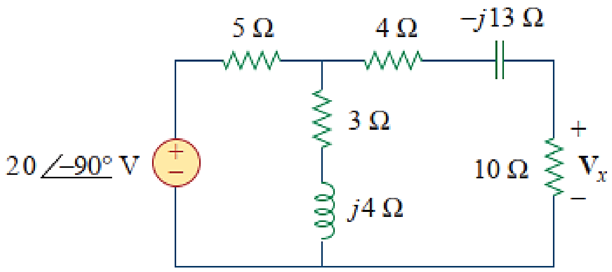


Fig.7

Solution:

We transform the voltage source to a current source and obtain the circuit in Fig.3 (a), where

$$\mathbf{I}_s = \frac{20\angle -90^\circ}{5} = 4\angle -90^\circ = -j4 \text{ A}$$

The parallel combination of 5-Ω resistance and $(3 + j4)$ impedance gives

$$\mathbf{Z}_1 = \frac{5(3 + j4)}{8 + j4} = 2.5 + j1.25\ \Omega$$

Converting the current source to a voltage source yields the circuit in Fig. 8(b), where,

$$\mathbf{V}_s = \mathbf{I}_s\mathbf{Z}_1 = -j4(2.5 + j1.25) = 5 - j10\ \text{V}$$

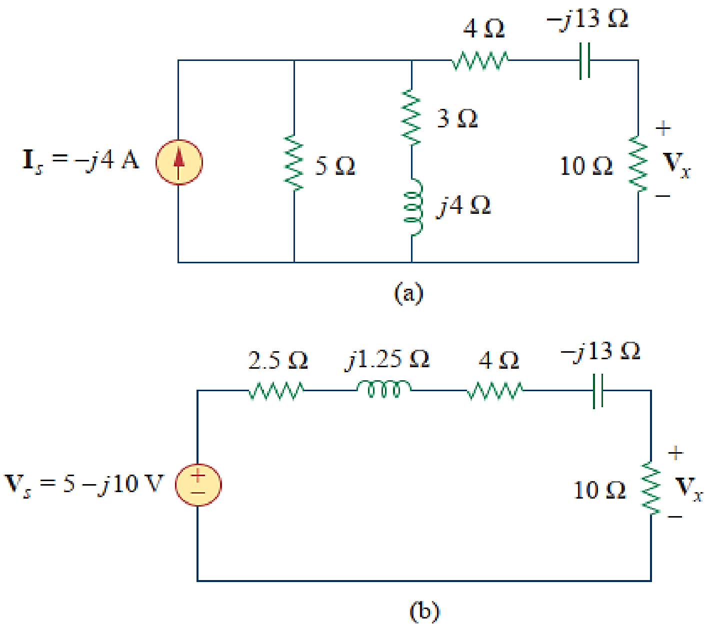


Fig. 8

By voltage division,

$$\mathbf{V}_x = \frac{10}{10 + 2.5 + j1.25 + 4 - j13}(5 - j10) = 5.519\underline{/-28^\circ}\ \text{V}$$

5. Thevenin and Norton Equivalent Circuits

Thevenin's and Norton's theorems are applied to ac circuits in the same way as they are to dc circuits. The only additional effort arises from the need to manipulate complex numbers. The frequency domain version of a Thevenin equivalent circuit is depicted in Fig. 9, where a linear circuit is replaced by a voltage source in series with an impedance.

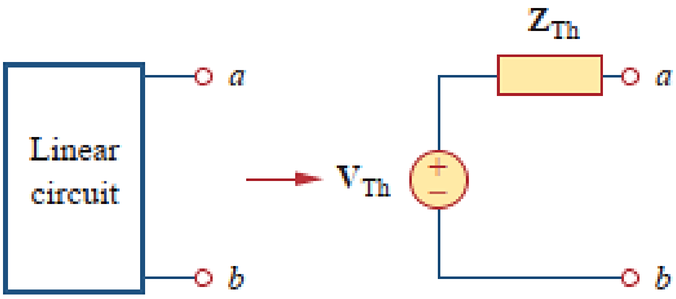


Fig. 9 Thevenin Equivalent Circuit

The Norton equivalent circuit is illustrated in Fig.10, where a linear circuit is replaced by a current source in parallel with an impedance.

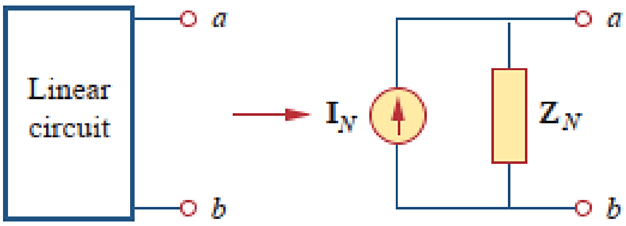


Fig. 10 Norton Equivalent Circuit

Keep in mind that the two equivalent circuits are related as

$$\boxed{\mathbf{V}_{Th} = \mathbf{Z}_N \mathbf{I}_N, \qquad \mathbf{Z}_{Th} = \mathbf{Z}_N}$$

just as in source transformation. $\mathbf{V}_{Th}$ is the **open-circuit voltage** while $\mathbf{I}_N$ is the **short-circuit current.**

If the circuit has sources operating **at different frequencies**, the Thevenin or Norton equivalent circuit must be determined at each frequency. This leads to entirely different equivalent circuits, one for each frequency, not one equivalent circuit with equivalent sources and equivalent impedances.

Steps to Apply Thevenin theorem

1. Remove that portion of the network across which the Thévenin equivalent circuit is to be found.
2. Mark (○, •, and so on) the terminals of the remaining two-terminal network.
3. Calculate Z_{Th} by first setting all voltage and current sources to zero (short circuit and open circuit, respectively) and then finding the resulting impedance between the two marked terminals.
4. Calculate E_{Th} by first replacing the voltage and current sources and then finding the open-circuit voltage between the marked terminals
5. Draw the Thévenin equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the Thévenin equivalent circuit.

Example 1:

Obtain the Thevenin equivalent at terminals a-b of the circuit in Fig.11

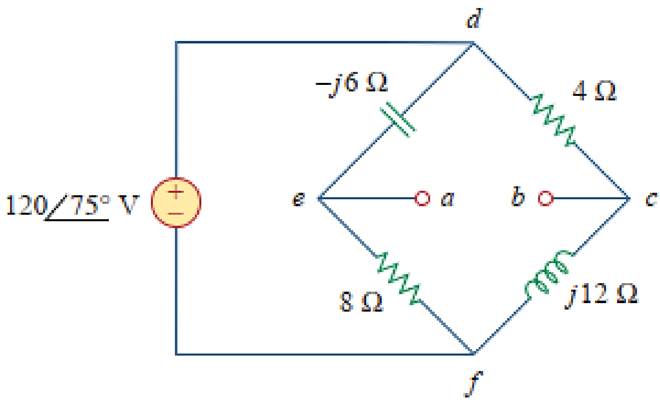


Fig.11

Solution:

We find Z_{th} by setting the voltage source to zero. As shown in Fig.12(a), the 8Ω resistance is now in parallel with $-j6$ the reactance, so that their combination gives

$$\mathbf{Z}_1 = -j6 \parallel 8 = \frac{-j6 \times 8}{8 - j6} = 2.88 - j3.84\ \Omega$$

Similarly, the 4Ωresistance is in parallel with the j12 reactance, and their combination gives

$$\mathbf{Z}_2 = 4 \parallel j12 = \frac{j12 \times 4}{4 + j12} = 3.6 + j1.2\ \Omega$$

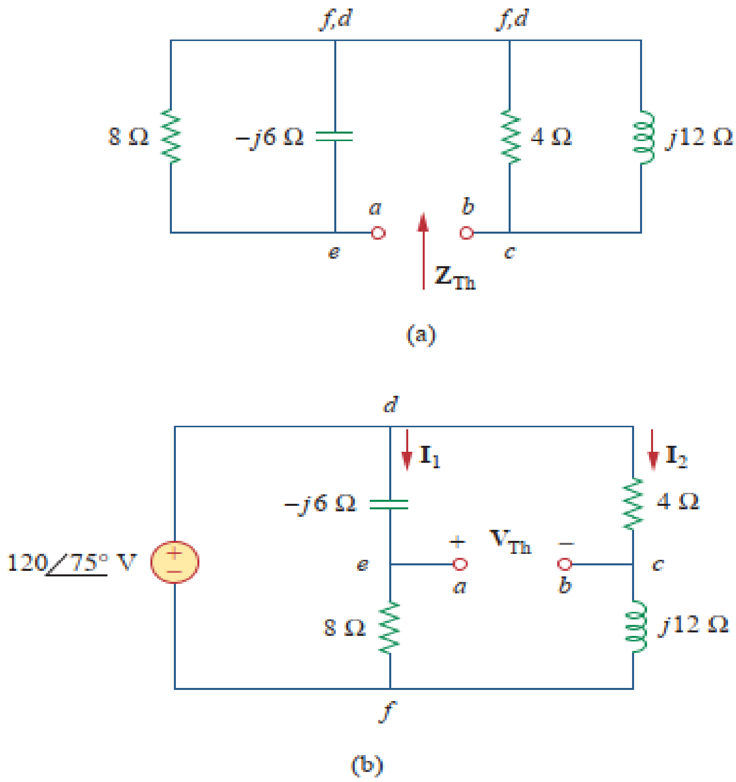


Fig.12 (a) finding Z_{Th}, (b) finding V_{Th}

The Thevenin impedance is the series combination of z_1 and z_2

that is,

$$\mathbf{Z}_{Th} = \mathbf{Z}_1 + \mathbf{Z}_2 = 6.48 - j2.64\ \Omega$$

To find V_{Th} consider the circuit in Fig. 12(b). Currents I_1 and I_2 are obtained as

$$\mathbf{I}_1 = \frac{120\angle 75^\circ}{8 - j6}\ \text{A}, \qquad \mathbf{I}_2 = \frac{120\angle 75^\circ}{4 + j12}\ \text{A}$$

Applying KVL around loop *bcdeab* in Fig. 4(b) gives

$$\mathbf{V}_{Th} - 4\mathbf{I}_2 + (-j6)\mathbf{I}_1 = 0$$

or

$$\begin{aligned}\mathbf{V}_{Th} = 4\mathbf{I}_2 + j6\mathbf{I}_1 &= \frac{480\angle 75^\circ}{4 + j12} + \frac{720\angle 75^\circ + 90^\circ}{8 - j6} \\ &= 37.95\angle 3.43^\circ + 72\angle 201.87^\circ \\ &= -28.936 - j24.55 = 37.95\angle 220.31^\circ\ \text{V}\end{aligned}$$

6. Norton theorem

Steps to Apply Norton theorem

1. Remove that portion of the network across which the Norton equivalent circuit is to be found.

2. Mark (○, •, and so on) the terminals of the remaining two-terminal network.

3. Calculate Z_N by first setting all voltage and current sources to zero (short circuit and open circuit, respectively) and then finding the resulting impedance between the two marked terminals.

4. Calculate I_N by first replacing the voltage and current sources and then finding the short-circuit current between the marked terminals.

5. Draw the Norton equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the Norton equivalent circuit.

Example 2:

Obtain current $\mathbf{I_o}$ in Fig. 13 using Norton's theorem.

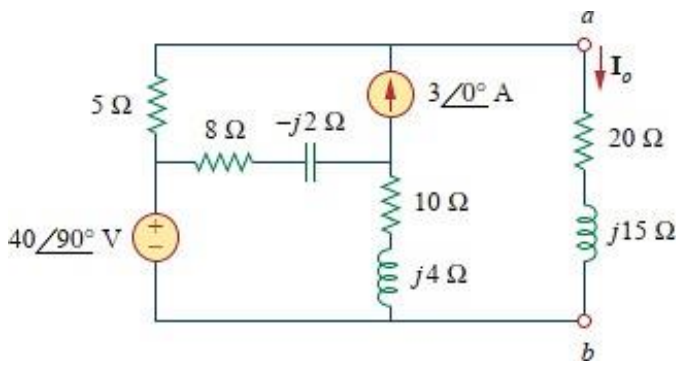


Fig.13

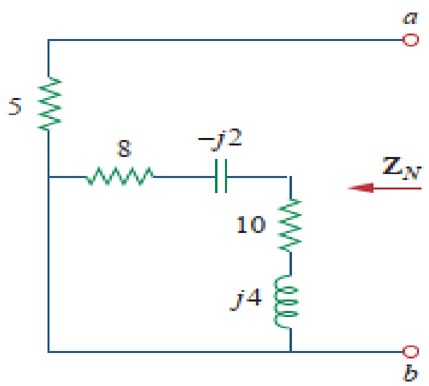
Solution:

Our first objective is to find the Norton equivalent at terminals a-b. z_N is found in the same way as z_{Th}. We set the sources to zero as shown in Fig. 14(a). As evident from the figure, the $8 - j2$ and $10 + j4$ impedances are short-circuited, so that

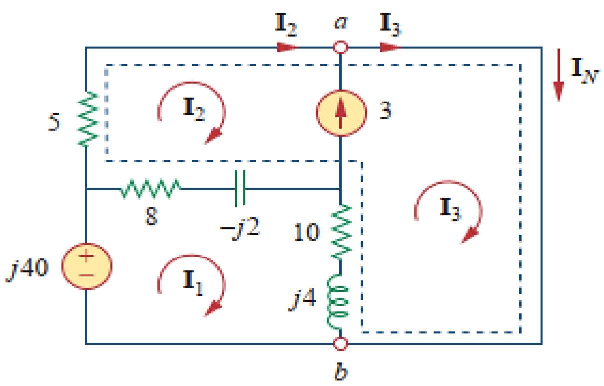
$$\mathbf{Z}_N = 5\ \Omega$$

To get I_N , we short-circuit terminals a-b as in Fig. 6(b) and apply mesh analysis. Notice that meshes 2 and 3 form a supermesh because of the current source linking them. For mesh 1,

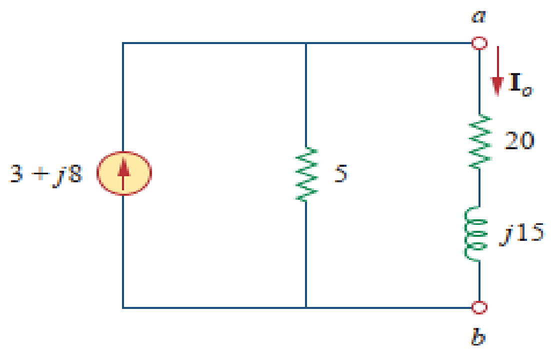
$$-j40 + (18 + j2)\mathbf{I}_1 - (8 - j2)\mathbf{I}_2 - (10 + j4)\mathbf{I}_3 = 0 \qquad (1)$$



(a)



(b)



(c)

Fig. 14: (a) finding Z_N, (b) finding V_N, (c) calculating I_o.

For the supermesh,

$$(13 - j2)\mathbf{I}_2 + (10 + j4)\mathbf{I}_3 - (18 + j2)\mathbf{I}_1 = 0 \qquad (2)$$

At node a, due to the current source between meshes 2 and 3,

$$\mathbf{I}_3 = \mathbf{I}_2 + 3 \qquad (3)$$

Adding Eqs. (1) and (2) gives

$$-j40 + 5\mathbf{I}_2 = 0 \qquad \Rightarrow \qquad \mathbf{I}_2 = j8$$

From Eq. (3),

$$\mathbf{I}_3 = \mathbf{I}_2 + 3 = 3 + j8$$

The Norton current is

$$\mathbf{I}_N = \mathbf{I}_3 = (3 + j8)\text{ A}$$

Figure 14(c) shows the Norton equivalent circuit along with the impedance at terminals a-b. By current division,

$$\mathbf{I}_o = \frac{5}{5 + 20 + j15}\mathbf{I}_N = \frac{3 + j8}{5 + j3} = 1.465\underline{/38.48^\circ}\text{ A}$$