Al-Mustaqbal University
College of Engineering & Technology
Building and Construction Techniques Engineering Dep.
Class (2nd)

MECHANICS OF MATERIALS

Lecture 1
Torsion

Lecturer:

Dr. Ammar Adil Albakri

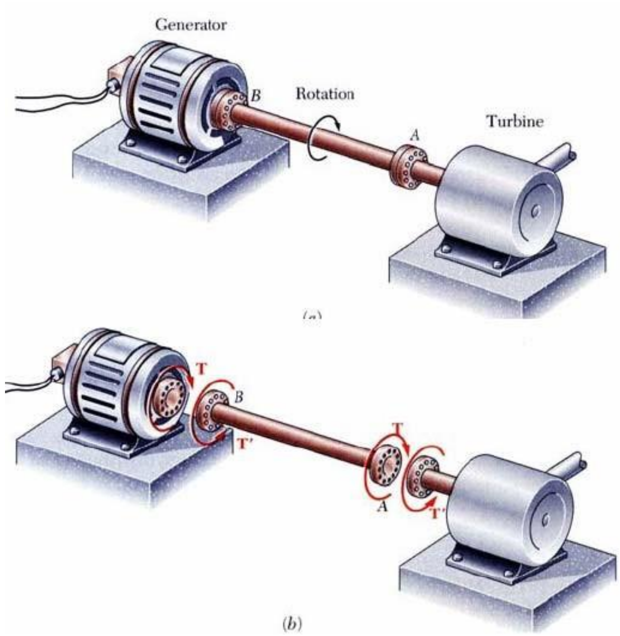
MECHANICS OF MATERIALS

Torsion

Ferdinand P. Beer, E. Russell Johnston, Jr., and John T. DeWolf

Torsional Loads on Circular Shafts

- Interested in stresses and strains of circular shafts subjected to twisting couples or *torques*
- Generator creates an equal and opposite torque *T*' • Shaft transmits the torque to the generator
- Turbine exerts torque *T* on the shaft



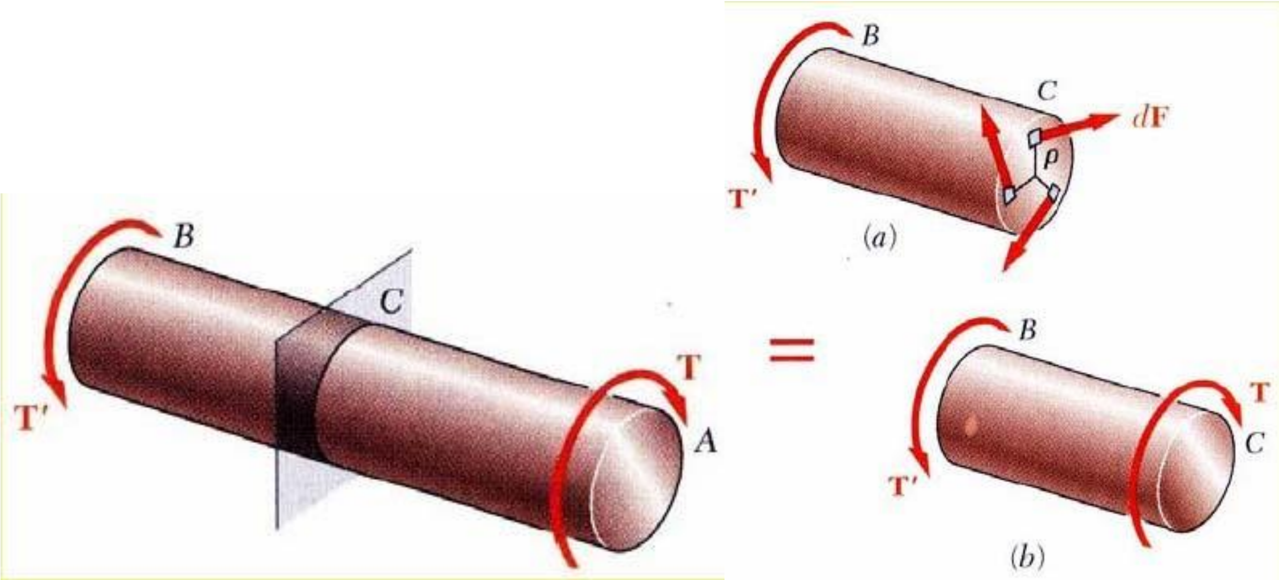
(b)

Net Torque Due to Internal Stresses

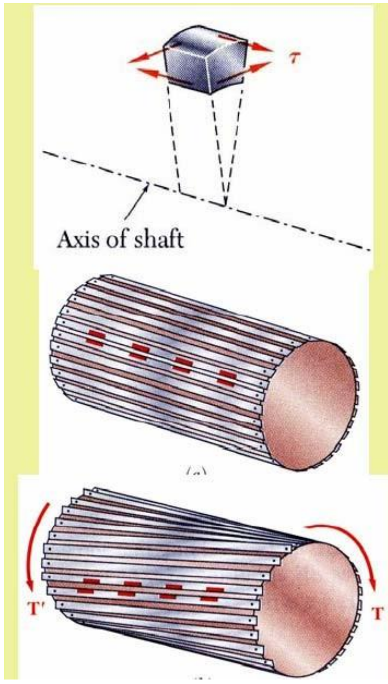
- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

$$T = \int \rho \, dF = \int \rho (\tau \, dA)$$

- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not
- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations
- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.



Axial Shear Components



- Torque applied to shaft produces shearing stresses on the faces perpendicular to the axis.
- Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft
- The existence of the axial shear components is demonstrated by considering a shaft made up of axial slats.

The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.

Shaft Deformations

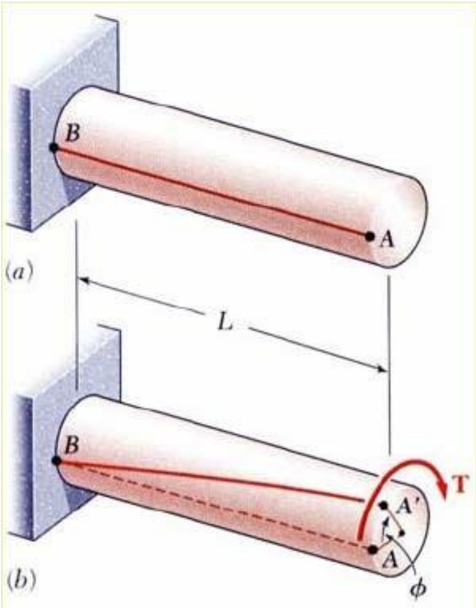
- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

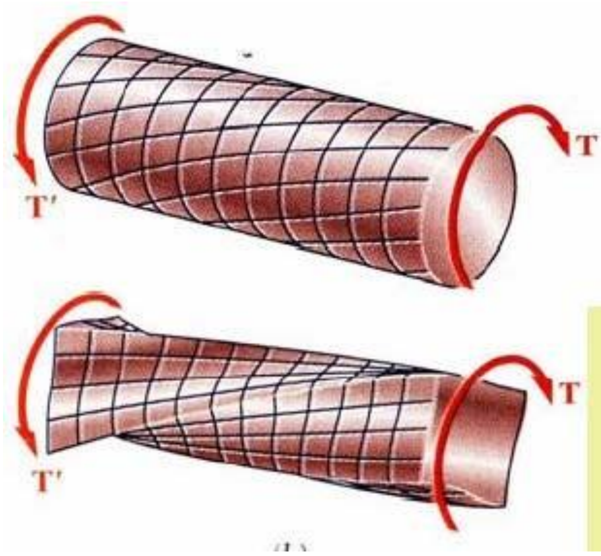
$$\phi \propto T$$

$$\phi \propto L$$

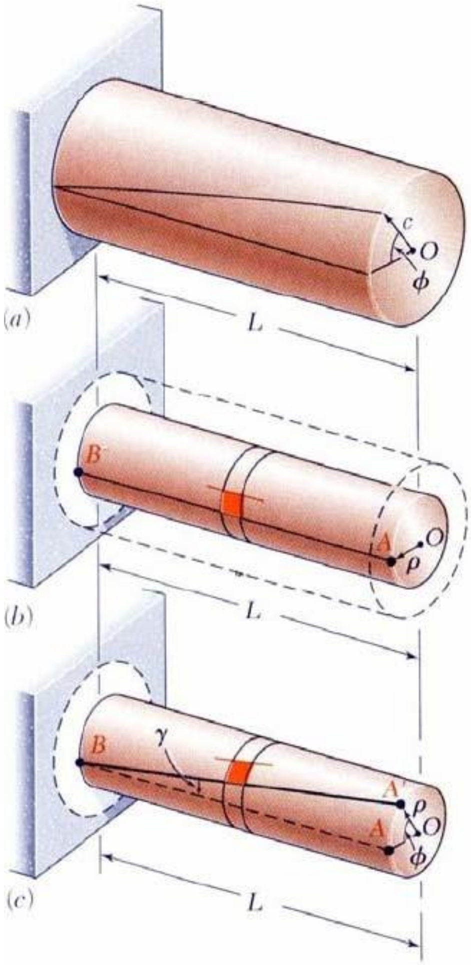
- When subjected to torsion, every cross- section of a circular shaft remains plane and undistorted.

- Cross- sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.
- Cross- sections of noncircular (nonaxisymmetric) shafts are distorted when subjected to torsion.





Shearing Strain



- • Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.
- • Since the ends of the element remain planar, the shear strain is equal to angle of twist.
- • It follows that

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

- • Shear strain is proportional to twist and radius

$$\gamma_{max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{max}$$

Stresses in Elastic Range

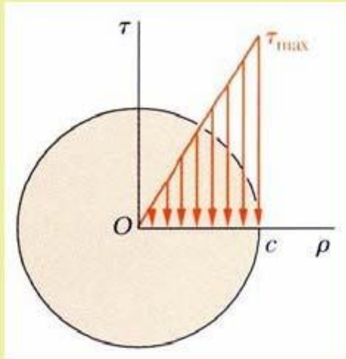
- Multiplying the previous equation by the shear modulus,

$$G\gamma = \frac{\rho}{c} G\gamma_{max}$$

From Hooke's Law, $\tau = G\gamma$, so

$$\tau = \frac{\rho}{c} \tau_{max}$$

- The shearing stress varies linearly with the radial position in the section.



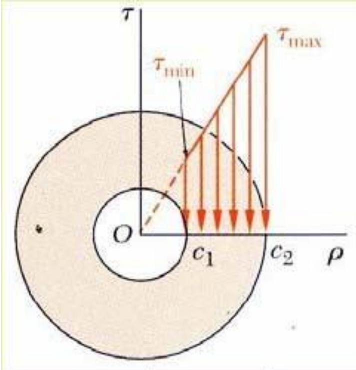
$$J = \frac{1}{2}\pi c^4$$

- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

$$T = \int \rho\tau \, dA = \frac{\tau_{max}}{c} \int \rho^2 \, dA = \frac{\tau_{max}}{c} J$$

- The results are known as the elastic torsion formulas,

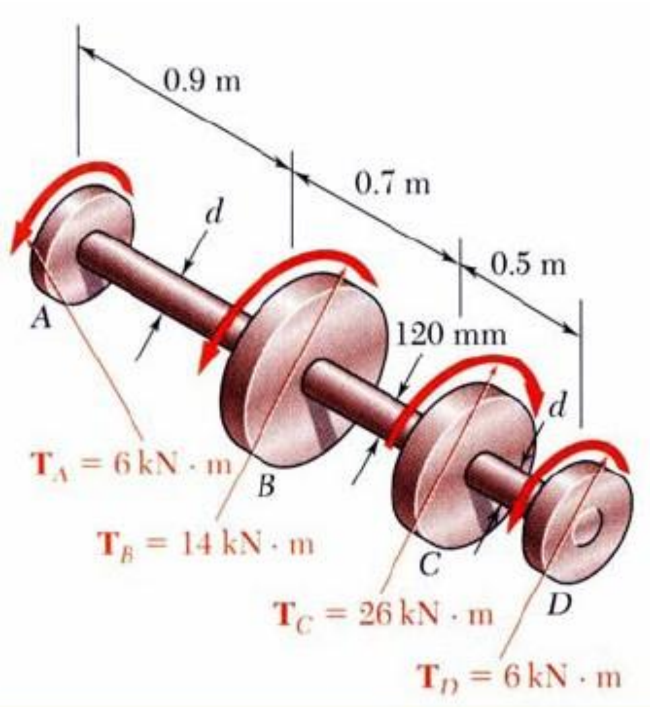
$$\tau_{max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$



$$J = \frac{1}{2}\pi\left(c_2^4 - c_1^4\right)$$

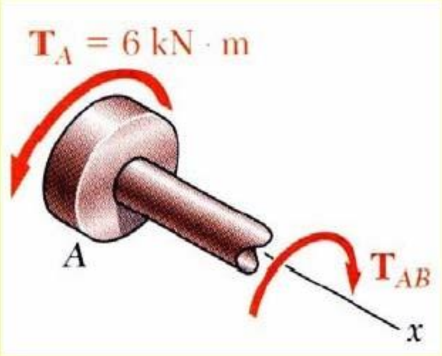
Sample Problem

Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid of diameter *d*. For the loading shown, determine (*a*) the minimum and maximum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.

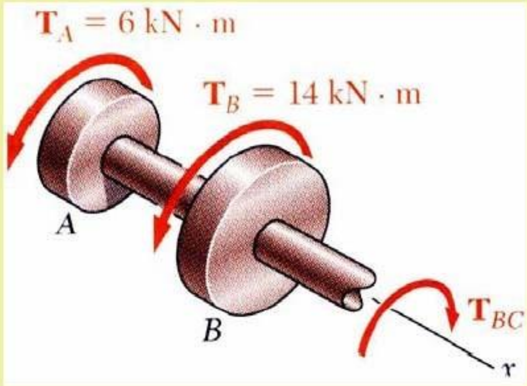


SOLUTION:

- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings

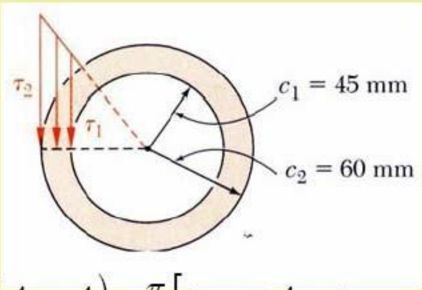


$$\sum M_x = 0 = (6\,\text{kN}\cdot\text{m}) - T_{AB}$$
$$T_{AB} = 6\,\text{kN}\cdot\text{m} = T_{CD}$$



$$\sum M_x = 0 = (6\,\text{kN}\cdot\text{m}) + (14\,\text{kN}\cdot\text{m}) - T_{BC}$$
$$T_{BC} = 20\,\text{kN}\cdot\text{m}$$

- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*



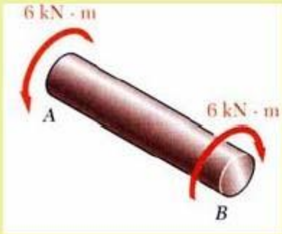
$$J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.060)^4 - (0.045)^4\right]$$
$$= 13.92\times 10^{-6}\,\text{m}^4$$

$$\tau_{max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\,\text{kN}\cdot\text{m})(0.060\,\text{m})}{13.92\times 10^{-6}\,\text{m}^4}$$
$$= 86.2\,\text{MPa}$$

$$\frac{\tau_{min}}{\tau_{max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{min}}{86.2\,\text{MPa}} = \frac{45\,\text{mm}}{60\,\text{mm}}$$
$$\tau_{min} = 64.7\,\text{MPa}$$

$$\tau_{max} = 86.2\,\text{MPa}$$
$$\tau_{min} = 64.7\,\text{MPa}$$

- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter



$$\tau_{max} = \frac{Tc}{J} = \frac{Tc}{\frac{\pi}{2}c^4} \qquad 65MPa = \frac{6\,\text{kN}\cdot\text{m}}{\frac{\pi}{2}c^3}$$

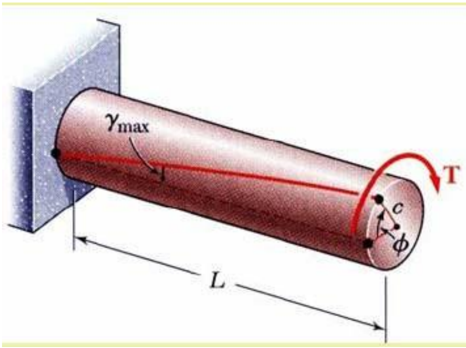
$$c = 38.9\times10^{-3}\,\text{m}$$

$$\boxed{d = 2c = 77.8\,\text{mm}}$$

Angle of Twist in Elastic Range

- Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{max} = \frac{c\phi}{L}$$



- In the elastic range, the shearing strain and shear are related by Hooke's Law,

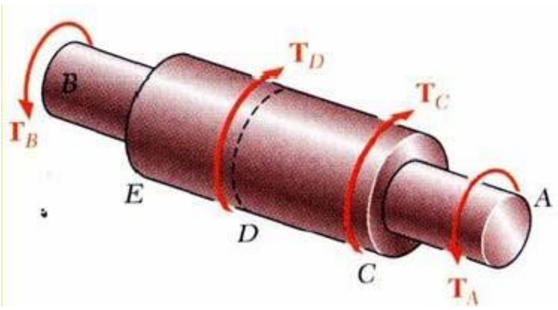
$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

- Equating the expressions for shearing strain and solving for the angle of twist,

$$\phi = \frac{TL}{JG}$$

- If the torsional loading or shaft cross- section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$



Design of Transmission Shafts

- Principal transmission shaft performance specifications are:
  - power
  - speed
- Designer must select shaft material and cross- section to meet performance specifications without exceeding allowable shearing stress.
- Determine torque applied to shaft at specified power and speed,

$$P = T\omega = 2\pi f T$$

$$T = \frac{P}{\omega} = \frac{P}{2\pi f}$$

- Find shaft cross- section which will not exceed the maximum allowable shearing stress,

$$\tau_{\max} = \frac{Tc}{J}$$

$$\frac{J}{c} = \frac{\pi}{2} c^3 = \frac{T}{\tau_{\max}} \quad \text{(solid shafts)}$$

$$\frac{J}{c_2} = \frac{\pi}{2c_2}\left(c_2^4 - c_1^4\right) = \frac{T}{\tau_{\max}} \quad \text{(hollow shafts)}$$

Important Notes

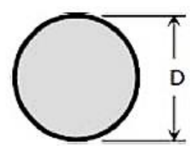
$$\frac{\tau}{r} = \frac{T}{J} = \frac{\theta G}{L}$$

$$\tau = \frac{Tr}{J} \qquad \theta = \frac{TL}{GJ}$$

For solid cylindrical shaft:

$$J = \frac{\pi}{32} D^4$$

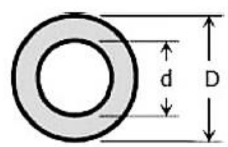
$$\tau_{max} = \frac{16T}{\pi D^3}$$



For hollow cylindrical shaft:

$$J = \frac{\pi}{32}(D^4 - d^4)$$

$$\tau_{max} = \frac{16TD}{\pi(D^4 - d^4)}$$



Where:

τ = Torsional shearing stress (N/m^2).

r= radius of section (m), (r= $D_o/2$) for sold or hollow shaft.

T= torque ($N.m$)

J= polar moment of inertia (m^4).

θ= angle of twist (rad). Chang it to degree multiply it by (180 / π) or (57.3).

L= shaft length (m).

G= modules of rigidity (N/m^2).

Power Transmitted by the Shaft:

$$P = T\omega = T\,2\pi f = T\,\frac{2\pi n}{60}$$

Where:

P= power transmitted (*watt*).

ω= angular velocity (*rad*).

n= revolution velocity (*r.p.m*).

f= frequency (*Hz* or *cycle/sec*).

SOLVED PROBLEMS IN TORSION

Problem No.1

A steel marine propeller shaft 14 in. in diameter and 18 ft long is used to transmit 5000 hp at 189 rpm. If G = 12 × 10^6 psi, determine the maximum shearing stress.

solution

$$T = \frac{P}{2\pi f} = \frac{5000(396\,000)}{2\pi(189)}$$

$$T = 1\,667\,337.5 \text{ lb·in}$$

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(1\,667\,337.5)}{\pi(14^3)}$$

$$\tau_{max} = 3094.6 \text{ psi}$$

Problem No.2

A steel shaft 3 ft long that has a diameter of 4 in. is subjected to a torque of 15 kip·ft. Determine the maximum shearing stress and the angle of twist. Use G = 12 × 10^6 psi.

Solution

$$\tau_{max} = \frac{16T}{\pi D^3} = \frac{16(15)(1000)(12)}{\pi(4^3)}$$

$$\tau_{max} = 14\,324 \text{ psi}$$

$$\tau_{max} = 14.3 \text{ ksi}$$

$$\theta = \frac{TL}{JG} = \frac{15(3)(1000)(12^2)}{\frac{1}{32}\pi(4^4)(12\times10^6)}$$

$$\theta = 0.0215 \text{ rad}$$

$$\theta = 1.23°$$

Problem No.3

What is the minimum diameter of a solid steel shaft that will not twist through more than 3° in a 6-m length when subjected to a torque of 12 kN·m? What maximum shearing stress is developed? Use G = 83 GPa.

Solution

$$\theta = \frac{TL}{JG}$$

$$3°\left(\frac{\pi}{180°}\right) = \frac{12(6)(1000^3)}{\frac{1}{32}\pi d^4(83000)}$$

$$d = 113.98 \text{ mm}$$

$$\tau_{max} = \frac{16T}{\pi d^3} = \frac{16(12)(1000^2)}{\pi(113.98^3)}$$

$$\tau_{max} = 41.27 \text{ MPa}$$

Problem No.4

Show that the hollow circular shaft whose inner diameter is half the outer diameter has a torsional strength equal to 15/16 of that of a solid shaft of the same outside diameter.

Solution

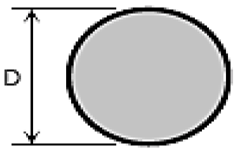
Hollow circular shaft:

$$\tau_{max\text{-}hollow} = \frac{16TD}{\pi(D^4 - d^4)}$$

$$= \frac{16TD}{\pi[D^4 - (\frac{1}{2}D)^4]}$$

$$= \frac{16TD}{\pi(\frac{15}{16}D^4)}$$

$$= \frac{16^2 T}{15\pi D^3}$$



Solid circular shaft:

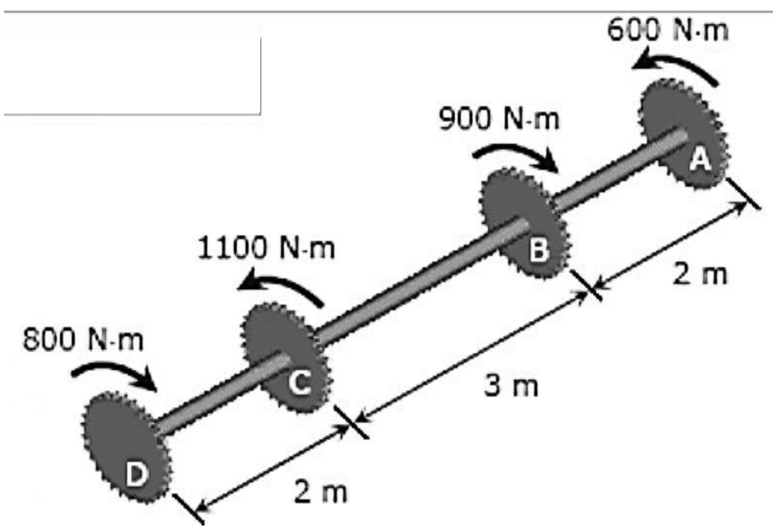
$$\tau_{max\text{-}solid} = \frac{16T}{\pi D^3}$$

$$= \frac{15}{16}\left[\frac{16^2 T}{15\pi D^3}\right]$$

$$= \tfrac{15}{16} \times \tau_{max\text{-}hollow} \quad \textit{ok!}$$

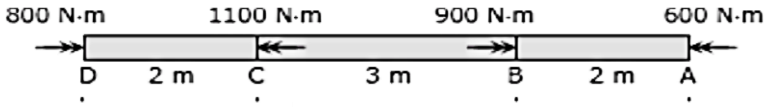
Problem No.5

An aluminum shaft with a constant diameter of 50 mm is loaded by torques applied to gears attached to it as shown in Fig. Using G = 28 GPa, determine the relative angle of twist of gear D relative to gear A.



Dr. Ammar Albakri

Solution



$$\theta = \frac{TL}{JG}$$

Rotation of D relative to A:

$$\theta_{D/A} = \frac{1}{JG}\sum TL$$

$$\theta_{D/A} = \frac{1}{\frac{1}{32}\pi(50^4)(28000)}\,[800(2) - 300(3) + 600(2)]\,(100^2)$$

$$\theta_{D/A} = 0.1106 \text{ rad}$$

$$\theta_{D/A} = 6.34°$$

Problem No.6

Determine the maximum torque that can be applied to a hollow circular steel shaft of 100-mm outside diameter and an 80-mm inside diameter without exceeding a shearing stress of 60 MPa or a twist of 0.5 deg/m. Use G = 83 GPa.

Solution

Based on maximum allowable shearing stress:

$$\tau_{max} = \frac{16TD}{\pi(D^4 - d^4)}$$

$$60 = \frac{16T(100)}{\pi(100^4 - 80^4)}$$

$$T = 6\,955\,486.14 \text{ N·mm}$$

$$T = 6\,955.5 \text{ N·m}$$

Based on maximum allowable angle of twist:

$$\theta = \frac{TL}{JG}$$

$$0.5°\left(\frac{\pi}{180°}\right) = \frac{T(1000)}{\frac{1}{32}\pi(100^4 - 80^4)(83\,000)}$$

$$T = 4\,198\,282.97 \text{ N·mm}$$

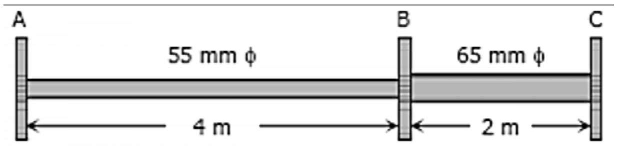
$$T = 4\,198.28 \text{ N·m}$$

Use the smaller torque, $T = 4\,198.28$ N·m

Dr. Ammar Albakri

Problem No.7

The steel shaft shown in Fig. rotates at 4 Hz with 35 kW taken off at A, 20 kW removed at B, and 55 kW applied at C. Using G = 83 GPa, find the maximum shearing stress and the angle of rotation of gear A relative to gear C.



Solution

$$T = \frac{P}{2\pi f}$$

$$T_A = \frac{-35(1000)}{2\pi(4)} = -1392.6 \text{ N·m}$$

$$T_B = \frac{-20(1000)}{2\pi(4)} = -795.8 \text{ N·m}$$

$$T_C = \frac{55(1000)}{2\pi(4)} = 2188.4 \text{ N·m}$$

Relative to C:

$$\tau_{max} = \frac{16T}{\pi d^3}$$

$$\tau_{AB} = \frac{16(1392.6)(1000)}{\pi(55^3)} = 42.63 \text{ MPa}$$

$$\tau_{BC} = \frac{16(2188.4)(1000)}{\pi(65^3)} = 40.58 \text{ MPa}$$

$$\therefore \quad \tau_{max} = \tau_{AB} = 42.63 \text{ MPa}$$

$$\theta = \frac{TL}{JG}$$

$$\theta_{A/C} = \frac{1}{G}\sum\frac{TL}{J}$$

$$\theta_{A/C} = \frac{1}{83\,000}\left[\frac{1392.6(4)}{\frac{1}{32}\pi(55^4)} + \frac{2188.4(2)}{\frac{1}{32}\pi(65^4)}\right](1000^2)$$

$$\theta_{A/C} = 0.104\,796\,585 \text{ rad}$$

$$\theta_{A/C} = 6.004°$$

Problem No. 8

A 5-m steel shaft rotating at 2 Hz has 70 kW applied at a gear that is 2 m from the left end where 20 kW are removed. At the right end, 30 kW are removed and another 20 kW leaves the shaft at 1.5 m from the right end. (a) Find the uniform shaft diameter so that the shearing stress will not exceed 60 MPa. (b) If a uniform shaft diameter of 100 mm is specified, determine the angle by which one end of the shaft lags behind the other end. Use G = 83 GPa.

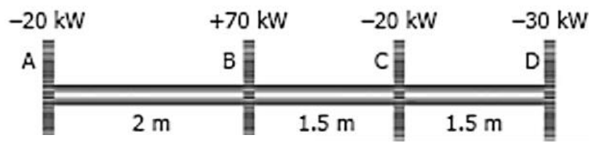
Solution

$$T = \frac{P}{2\pi f}$$

$$T_A = T_C = \frac{-20(1000)}{2\pi(2)} = -1591.55 \text{ N·m}$$

$$T_B = \frac{70(1000)}{2\pi(2)} = 5570.42 \text{ N·m}$$

$$T_D = \frac{-30(1000)}{2\pi(2)} = -2387.32 \text{ N·m}$$



Part (*a*)

$$\tau_{max} = \frac{16T}{\pi d^3}$$

For AB: $$60 = \frac{16(1591.55)(1000)}{\pi d^3}$$
$d = 51.3$ mm

For BC: $$60 = \frac{16(3978.87)(1000)}{\pi d^3}$$
$d = 69.6$ mm

For CD: $$60 = \frac{16(2387.32)(1000)}{\pi d^3}$$
$d = 58.7$ mm

Use **$d = 69.6$ mm**

Part (*b*)

$$\theta = \frac{TL}{JG}$$

$$\theta_{D/A} = \frac{1}{JG}\sum TL$$

$$\theta_{D/A} = \frac{1}{\frac{1}{32}\pi(100^4)(83000)}[-1591.55(2) + 3978.87(1.5) + 2387.32(1.5)]\,(1000^2)$$

$\theta_{D/A} = 0.007\,813$ rad

$\theta_{D/A} = \mathbf{0.448°}$

Dr. Ammar Albakri