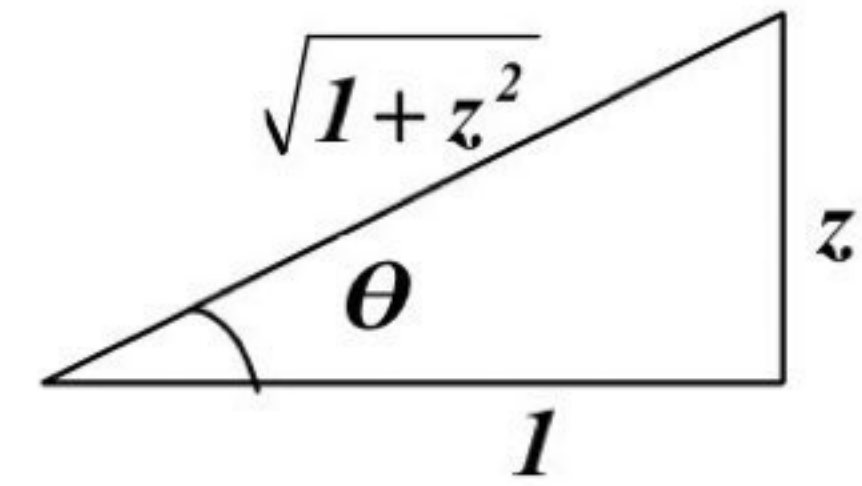


Sol.-

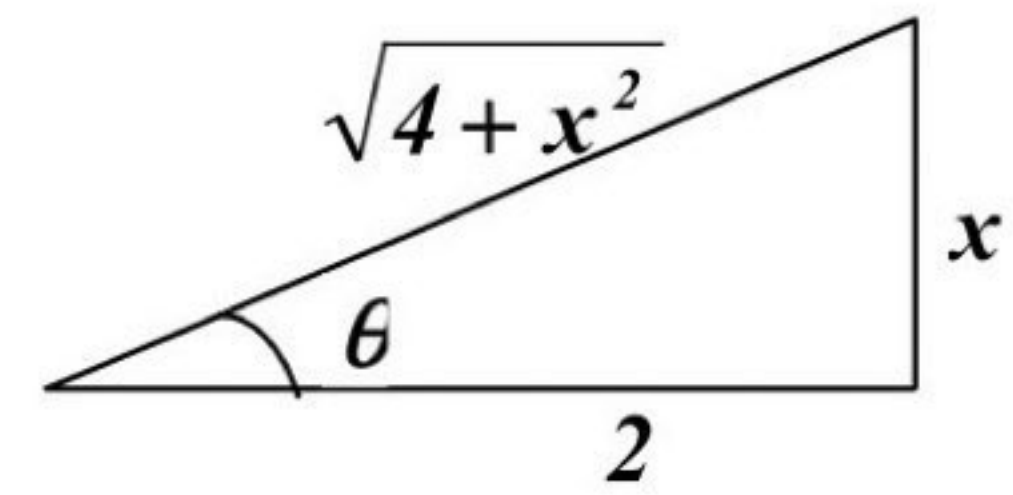
$$1) \text{ let } z = \tan \theta \Rightarrow dz = \sec^2 \theta \cdot d\theta \quad \tan \theta = \frac{z}{1}$$

$$\begin{aligned} \int \frac{z^5 dz}{\sqrt{1+z^2}} &= \int \frac{\tan^5 \theta \cdot \sec^2 \theta d\theta}{\sqrt{1+\tan^2 \theta}} = \int \tan^5 \theta \cdot \sec \theta d\theta \\ &= \int \tan \theta \cdot \sec \theta (\sec^2 \theta - 1)^2 d\theta \\ &= \int \sec^4 \theta (\tan \theta \cdot \sec \theta d\theta) - 2 \int \sec^2 \theta (\tan \theta \cdot \sec \theta d\theta) + \int \tan \theta \cdot \sec \theta d\theta \\ &= \frac{1}{5} \sec^5 \theta - \frac{2}{3} \sec^3 \theta + \sec \theta + c \\ &= \frac{1}{5} (\sqrt{1+z^2})^5 - \frac{2}{3} (\sqrt{1+z^2})^3 + \sqrt{1+z^2} + c \end{aligned}$$



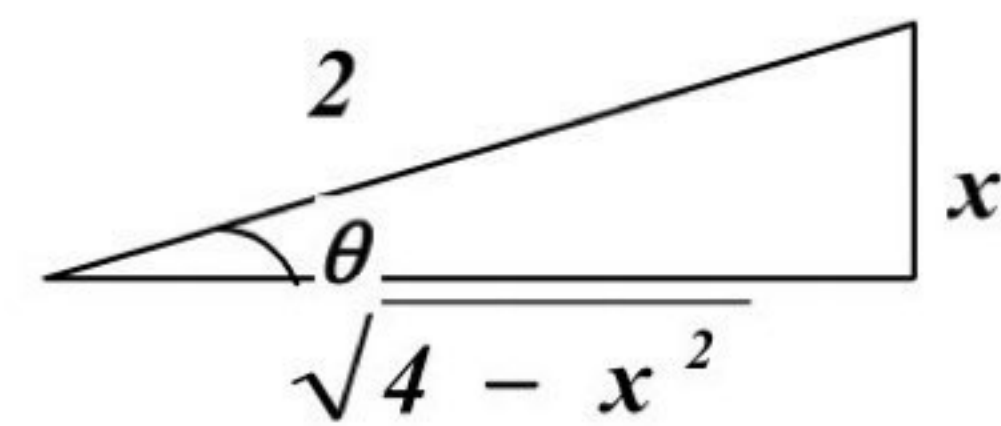
$$2) \text{ let } x = 2 \tan \theta \Rightarrow dx = 2 \sec^2 \theta \cdot d\theta \quad \tan \theta = \frac{x}{2}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2 \sec^2 \theta d\theta}{\sqrt{4+4 \tan^2 \theta}} = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + c \\ &= \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + c \\ &= \ln |\sqrt{4+x^2} + x| + c' \quad \text{where } c' = c - \ln 2 \end{aligned}$$



$$3) \text{ let } x = 2 \sin \theta \Rightarrow dx = 2 \cos \theta \cdot d\theta$$

$$\begin{aligned} \int \frac{dx}{4-x^2} &= \int \frac{2 \cos \theta d\theta}{4-4 \sin^2 \theta} = \frac{1}{2} \int \frac{d\theta}{\cos \theta} = \frac{1}{2} \int \sec \theta d\theta \\ &= \frac{1}{2} \ln |\sec \theta + \tan \theta| + c \\ &= \frac{1}{2} \ln \left| \frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}} \right| + c \\ &= \frac{1}{2} \ln \left| \frac{2+x}{\sqrt{(2-x)(2+x)}} \right| + c = \frac{1}{2} \ln \left| \sqrt{\frac{2+x}{2-x}} \right| + c = \frac{1}{4} \ln \left| \frac{2+x}{2-x} \right| + c \end{aligned}$$



6-5- Partial fractions:

Success in separating $\frac{f(x)}{g(x)}$ into a sum of partial fractions hinges on two things:-

- 1- The degree of $f(x)$ must be less than the degree of $g(x)$. (If this is not case, we first perform a long division, and then work with the remainder term).
- 2- The factors of $g(x)$ must be known. If these two conditions are met we can carry out the following steps:

Step I - let $x - r$ be a linear factor of $g(x)$. Suppose $(x - r)^m$ is the highest power of $(x - r)$ that divides $g(x)$. Then assign the sum of m partial factors to this factor, as follows:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \dots + \frac{A_m}{(x - r)^m}$$

Do this for each distinct linear factor of $f(x)$.

Step II - let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$. Suppose $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1 x + C_1}{x^2 + px + q} + \frac{B_2 x + C_2}{(x^2 + px + q)^2} + \dots + \frac{B_n x + C_n}{(x^2 + px + q)^n}$$

Do this for each distinct linear factor of $g(x)$.

Step III - set the original fraction $\frac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the sums in decreasing powers of x .

Step IV - equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EX-8 – Evaluate the following integrals:

$$1) \int \frac{2x+5}{x^2-9} dx$$

$$4) \int \frac{\sin x \, dx}{\cos^2 x - 5 \cos x + 4}$$

$$2) \int \frac{x \, dx}{x^2+4x+3}$$

$$5) \int \frac{2x^2-3x+2}{(x-1)^2(x-2)} dx$$

$$3) \int \frac{x^3-x}{(x^2+1) \cdot (x-1)^2} dx$$

$$6) \int \frac{x^3+4x^2}{x^2+4x+3} dx$$

Sol.-

$$1) \int \frac{2x+5}{x^2-9} dx = \int \frac{2x+5}{(x-3) \cdot (x+3)} dx$$

$$\frac{2x+5}{(x-3) \cdot (x+3)} = \frac{A}{x-3} + \frac{B}{x+3} \Rightarrow 2x+5 = A(x+3) + B(x-3)$$

$$\text{at } x=3 \Rightarrow 6A=6+5 \Rightarrow A=\frac{11}{6}$$

$$\text{at } x=-3 \Rightarrow -6B=-6+5 \Rightarrow B=\frac{1}{6}$$

$$\int \frac{2x+5}{x^2-9} dx = \int \left(\frac{11/6}{x-3} + \frac{1/6}{x+3} \right) dx = \frac{11}{6} \ln(x-3) + \frac{1}{6} \ln(x+3) + c$$

$$2) \int \frac{x \, dx}{x^2+4x+3} = \int \frac{x \, dx}{(x+3)(x+1)}$$

$$\frac{x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} \Rightarrow x = A(x+1) + B(x+3)$$

$$\text{at } x=-3 \Rightarrow A=\frac{3}{2} \text{ and at } x=-1 \Rightarrow B=-\frac{1}{2}$$

$$\int \frac{x \, dx}{x^2+4x+3} = \int \left(\frac{3/2}{x+3} + \frac{-1/2}{x+1} \right) dx = \frac{3}{2} \ln(x+3) - \frac{1}{2} \ln(x+1) + c$$

$$3) \int \frac{x^3 - x}{(x^2 + 1)(x - 1)^2} dx = \int \frac{x(x - 1)(x + 1)}{(x^2 + 1)(x - 1)^2} dx = \int \frac{x^2 + x}{(x^2 + 1)(x - 1)} dx$$

$$\frac{x^2 + x}{(x^2 + 1)(x - 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} \Rightarrow x^2 + x = (Ax + B)(x - 1) + C(x^2 + 1)$$

$$x^2 + x = (A + C)x^2 + (-A + B)x + (-B + C)$$

$$\left. \begin{array}{l} A + C = 1 \text{(1)} \\ -A + B = 1 \text{(2)} \\ -B + C = 0 \text{(3)} \end{array} \right\} \Rightarrow A = 0, B = 1, C = 1$$

$$\int \frac{x^3 - x}{(x^2 + 1)(x - 1)^2} dx = \int \left(\frac{1}{x^2 + 1} + \frac{1}{x - 1} \right) dx = \tan^{-1} x + \ln(x - 1) + c$$

$$4) \text{ let } y = \cos x \Rightarrow dy = -\sin x dx$$

$$\int \frac{\sin x dx}{\cos^2 x - 5 \cos x + 4} = -\int \frac{dy}{y^2 - 5y + 4} = -\int \frac{dy}{(y - 4)(y - 1)}$$

$$\frac{dy}{(y - 4)(y - 1)} = \frac{A}{y - 4} + \frac{B}{y - 1} \Rightarrow 1 = A(y - 1) + B(y - 4)$$

$$\text{at } y = 4 \Rightarrow A = \frac{1}{3} \text{ and at } y = 1 \Rightarrow B = -\frac{1}{3}$$

$$\int \frac{\sin x dx}{\cos^2 x - 5 \cos x + 4} = -\int \left(\frac{1/3}{y - 4} + \frac{-1/3}{y - 1} \right) dy$$

$$= -\frac{1}{3} \ln(y - 4) + \frac{1}{3} \ln(y - 1) + c = -\frac{1}{3} \ln(\cos x - 4) + \frac{1}{3} \ln(\cos x - 1) + c$$

$$5) \frac{2x^2 - 3x + 2}{(x - 1)^2(x - 2)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x - 2}$$

$$2x^2 - 3x + 2 = A(x - 1)(x - 2) + B(x - 2) + C(x - 1)^2$$

$$\left. \begin{array}{l} A + C = 2 \text{(1)} \\ -3A + B - 2C = -3 \text{(2)} \\ 2A - 2B + C = 2 \text{(3)} \end{array} \right\} \Rightarrow A = -2, B = -1, C = 4$$

$$\int \frac{2x^2 - 3x + 2}{(x - 1)^2(x - 2)} dx = \int \left(\frac{-2}{x - 1} + \frac{1}{(x - 1)^2} + \frac{4}{x - 2} \right) dx$$

$$= -2 \ln(x - 1) + \frac{1}{x - 1} + 4 \ln(x - 2) + C$$

$$6) \frac{x^3 + 4x^2}{x^2 + 4x + 3} = x - \frac{3x}{(x+3)(x+1)} \quad \begin{array}{r} x^2 + 4x + 3 \overline{) x^3 + 4x^2} \\ \underline{ + x^3 + 4x^2 + 3x} \\ -3x \end{array}$$

$$\frac{3x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} \Rightarrow 3x = A(x+1) + B(x+3)$$

$$\text{at } x = -3 \Rightarrow A = \frac{9}{2} \text{ and at } x = -1 \Rightarrow B = -\frac{3}{2}$$

$$\begin{aligned} \int \frac{x^3 + 4x^2}{x^2 + 4x + 3} dx &= \int \left(x - \frac{9/2}{x+3} + \frac{3/2}{x+1} \right) dx \\ &= \frac{x^2}{2} - \frac{9}{2} \ln(x+3) - \frac{3}{2} \ln(x+1) + c \end{aligned}$$

6-6- Rational functions of $\sin x$ and $\cos x$, and other trigonometric integrals:

We assume that $z = \tan \frac{x}{2}$ then $x = 2 \tan^{-1} z$ and $dx = \frac{2}{1+z^2} dz$

Since

$$\begin{aligned} \cos^2 \frac{x}{2} &= \frac{1 + \cos x}{2} \Rightarrow \cos x = 2 \cos^2 \frac{x}{2} - 1 = \frac{2}{\sec^2 \frac{x}{2}} - 1 \\ &= \frac{2}{\tan^2 \frac{x}{2} + 1} - 1 = \frac{2}{z^2 + 1} - 1 \Rightarrow \cos x = \frac{1 - z^2}{1 + z^2} \end{aligned}$$

Since

$$\begin{aligned} \sin x &= 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2} = \frac{2 \sin \frac{x}{2}}{\cos \frac{x}{2}} \cdot \cos^2 \frac{x}{2} = 2 \tan \frac{x}{2} \cdot \frac{1}{\sec^2 \frac{x}{2}} \\ &= 2 \tan \frac{x}{2} \cdot \frac{1}{\tan^2 \frac{x}{2} + 1} \Rightarrow \sin x = \frac{2z}{1 + z^2} \end{aligned}$$

EX-9 – Evaluate:

$$1) \int \frac{dx}{1 + \sin x + \cos x}$$

$$4) \int \frac{3 dx}{2 + 4 \sin x}$$

$$2) \int \frac{dx}{\sin x + \tan x}$$

$$5) \int \sec x dx$$

$$3) \int \frac{dx}{2 + \sin x}$$

$$6) \int \frac{\cos x dx}{1 - \cos x}$$

Sol.-

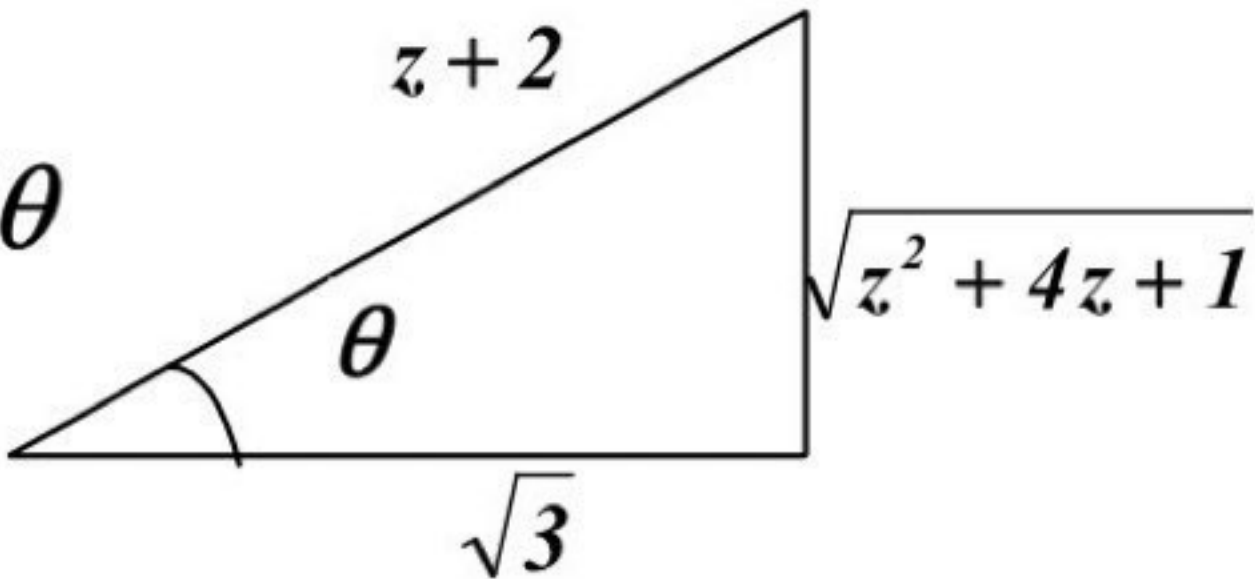
$$\begin{aligned} 1) \int \frac{dx}{1 + \sin x + \cos x} &= \int \frac{\frac{2}{1+z^2} dz}{1 + \frac{2z}{1+z^2} + \frac{1-z^2}{1+z^2}} = \int \frac{dz}{1+z} \\ &= \ln|1+z| + c = \ln\left|1 + \tan \frac{x}{2}\right| + c \end{aligned}$$

$$\begin{aligned} 2) \int \frac{dx}{\sin x + \tan x} &= \int \frac{\frac{2}{1+z^2} dz}{\frac{2z}{1+z^2} + \frac{2z}{1-z^2}} = \frac{1}{2} \int \left(\frac{1}{z} - z\right) dz \\ &= \frac{1}{2} \left[\ln z - \frac{z^2}{2} \right] + c = \frac{1}{2} \left[\ln \tan \frac{x}{2} - \frac{1}{2} \tan^2 \frac{x}{2} \right] + c \end{aligned}$$

$$\begin{aligned} 3) \int \frac{dx}{2 + \sin x} &= \int \frac{\frac{2}{1+z^2} dz}{2 + \frac{2z}{1+z^2}} = \int \frac{dz}{z^2 + z + 1} = \int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}} \\ \text{let } z + \frac{1}{2} &= \frac{\sqrt{3}}{2} \tan \theta \Rightarrow dz = \frac{\sqrt{3}}{2} \sec^2 \theta d\theta \\ \int \frac{dx}{2 + \sin x} &= \int \frac{\frac{\sqrt{3}}{2} \sec^2 \theta d\theta}{\frac{3}{4} \tan^2 \theta + \frac{3}{4}} = \frac{2}{\sqrt{3}} \int d\theta = \frac{2}{\sqrt{3}} \theta + c \\ &= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2z+1}{\sqrt{3}} \right) + c = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan \frac{x}{2} + 1}{\sqrt{3}} \right) + c \end{aligned}$$

$$\begin{aligned}
 4) \int \frac{3 \, dx}{2 + 4 \sin x} &= \frac{3}{2} \int \frac{dx}{1 + 2 \sin x} = \frac{3}{2} \int \frac{\frac{2}{z^2 + 1} dz}{1 + 2 \frac{2z}{z^2 + 1}} = 3 \int \frac{dz}{z^2 + 4z + 1} \\
 &= 3 \int \frac{dz}{(z + 2)^2 - 3} = \int \frac{dz}{\left(\frac{z + 2}{\sqrt{3}}\right)^2 - 1}
 \end{aligned}$$

$$\text{let } \frac{z + 2}{\sqrt{3}} = \sec \theta \Rightarrow dz = \sqrt{3} \sec \theta \cdot \tan \theta \, d\theta$$

$$\int \frac{3 \, dx}{2 + 4 \sin x} = \int \frac{\sqrt{3} \sec \theta \cdot \tan \theta \, d\theta}{\sec^2 \theta - 1} = \sqrt{3} \int \frac{\sec \theta}{\tan \theta} d\theta$$


$$= \sqrt{3} \int \csc \theta \, d\theta = -\sqrt{3} \ln |\csc \theta + \cot \theta| + c$$

$$= -\sqrt{3} \ln \left| \frac{z + 2}{\sqrt{z^2 + 4z + 1}} + \frac{\sqrt{3}}{\sqrt{z^2 + 4z + 1}} \right| + c = -\sqrt{3} \ln \left| \frac{\tan \frac{x}{2} + 2 + \sqrt{3}}{\sqrt{\tan^2 \frac{x}{2} + 4 \tan \frac{x}{2} + 1}} \right| + c$$

$$5) \int \sec x \, dx = \int \frac{1 + z^2}{1 - z^2} \cdot \frac{2}{1 + z^2} dz = 2 \int \frac{1}{(1 - z)(1 + z)} dz$$

$$\frac{1}{(1 - z)(1 + z)} = \frac{A}{1 - z} + \frac{B}{1 + z} \Rightarrow A(1 + z) + B(1 - z) = 1$$

$$\text{at } z = 1 \Rightarrow A = \frac{1}{2} \text{ and at } z = -1 \Rightarrow B = \frac{1}{2}$$

$$\int \sec x \, dx = 2 \int \left(\frac{1/2}{1 - z} + \frac{1/2}{1 + z} \right) dz = -\ln(1 - z) + \ln(1 + z) + c$$

$$= \ln \left(1 + \tan \frac{x}{2} \right) - \ln \left(1 - \tan \frac{x}{2} \right) + c = \ln \left| \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right| + c$$

$$\text{By substituting } \tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}} \text{ implies}$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + c$$

$$6) \int \frac{\cos x \, dx}{1 - \cos x} = \int \frac{\frac{1 - z^2}{1 + z^2}}{1 - \frac{1 - z^2}{1 + z^2}} \cdot \frac{2}{1 + z^2} dz = \int \frac{1 - z^2}{(1 + z^2) z^2} dz$$

$$\frac{1 - z^2}{(1 + z^2) z^2} = \frac{A}{z} + \frac{B}{z^2} + \frac{Cz + D}{1 + z^2} \Rightarrow$$

$$Az + Az^3 + B + Bz^2 + Cz^3 + Dz^2 = 1 - z^2$$

$$\left. \begin{array}{l} A + C = 0 \quad \dots\dots(1) \\ B + D = -1 \quad \dots(2) \\ A = 0 \quad \dots\dots\dots(3) \\ B = 1 \quad \dots\dots\dots(4) \end{array} \right\} \Rightarrow A = 0, B = 1, C = 0, D = -2$$

$$\int \frac{\cos x \, dx}{1 - \cos x} = \int \left(\frac{1}{z^2} - \frac{2}{z^2 + 1} \right) dz = -\frac{1}{z} - 2 \tan^{-1} z + c$$

$$= -\frac{1}{\tan \frac{x}{2}} - 2 \cdot \frac{x}{2} + c = -\cot \frac{x}{2} - x + c$$

Problems – 6

Evaluate the following integrals:

- 1) $\int \frac{x^3}{x-1} dx$ (ans.: $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x + \ln(x-1) + c$)
- 2) $\int \frac{3x+2}{3x-1} dx$ (ans.: $x + \ln(3x-1) + c$)
- 3) $\int x^2 \cdot e^{-x} dx$ (ans.: $-e^{-x}(x^2 + 2x + 2) + c$)
- 4) $\int x \cdot \sin x^2 dx$ (ans.: $-\frac{1}{2}\cos x^2 + c$)
- 5) $\int \sqrt{x^2-1} dx$ (ans.: $\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln|x + \sqrt{x^2+1}| + c$)
- 6) $\int \frac{3x+13}{(5x-1)(7x+2)} dx$ (ans.: $\frac{4}{5}\ln|5x-1| - \frac{5}{7}\ln|7x+2| + c$)
- 7) $\int \frac{2x-3}{(x-1)(x-2)(x+3)} dx$ (ans.: $\frac{1}{4}\ln|x-1| + \frac{1}{5}\ln|x-2| - \frac{9}{20}\ln|x+3| + c$)
- 8) $\int \frac{dx}{x^4-1}$ (ans.: $\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + c$)
- 9) $\int \ln x dx$ (ans.: $x \cdot \ln x - x + c$)
- 10) $\int \tan^{-1} x dx$ (ans.: $x \cdot \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$)
- 11) $\int x \cdot \ln x dx$ (ans.: $\frac{x^2}{2}\ln x - \frac{x^2}{4} + c$)
- 12) $\int x \cdot \tan^{-1} x dx$ (ans.: $\frac{x^2}{2}\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + c$)
- 13) $\int x^2 \cdot \cos ax dx$ (ans.: $\frac{x^2}{a}\sin ax + \frac{2x}{a^2}\cos ax - \frac{2}{a^3}\sin ax + c$)
- 14) $\int \sin(\ln x) dx$ (ans.: $\frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + c$)
- 15) $\int \ln(a^2 + x^2) dx$ (ans.: $x \cdot \ln(a^2 + x^2) - 2x + 2a \tan^{-1} \frac{x}{a} + c$)

$$16) \int x \cdot \sin^{-1} x \, dx$$

$$(ans.: \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + c)$$

$$17) \int \cos^4 x \, dx$$

$$(ans.: \frac{3x}{8} + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c)$$

$$18) \int \cos^{\frac{2}{3}} x \cdot \sin^5 x \, dx$$

$$(ans.: -\frac{3}{5} \cos^{\frac{5}{3}} x + \frac{6}{11} \cos^{\frac{11}{3}} x - \frac{3}{17} \cos^{\frac{17}{3}} x + c)$$

$$19) \int x \cdot \sin x \, dx$$

$$(ans.: -x \cdot \cos x + \sin x + c)$$

$$20) \int x^2 \sqrt{1-x} \, dx$$

$$(ans.: -\frac{2}{105} \sqrt{(1-x)^3} (15x^2 + 12x + 8) + c)$$

$$21) \int \sin^2 x \cdot \cos^2 x \, dx$$

$$(ans.: \frac{1}{32} (4x - \sin 4x) + c)$$

$$22) \int \sec^3 x \cdot \tan^2 x \, dx$$

$$(ans.: \frac{1}{4} \sec^3 x \cdot \tan x - \frac{1}{8} \sec x \cdot \tan x - \frac{1}{8} \ln |\sec x + \tan x| + c)$$

$$23) \int x (\cos^3 x^2 - \sin^3 x^2) dx$$

$$(ans.: \frac{1}{2} \sin x^2 - \frac{1}{6} \sin^3 x^2 + \frac{1}{2} \cos x^2 - \frac{1}{6} \cos^3 x^2 + c)$$

$$24) \int \frac{dx}{\sqrt{x} \sqrt{1-x}}$$

$$(ans.: 2 \sin^{-1} \sqrt{x} + c)$$

$$25) \int \frac{dx}{\sqrt{x} \cdot (1 + \sqrt{x})}$$

$$(ans.: 2 \ln(1 + \sqrt{x}) + c)$$

$$26) \int \frac{dx}{x \sqrt{2-3 \ln^2 \sqrt{x}}}$$

$$(ans.: \frac{2}{\sqrt{3}} \sin^{-1} (\frac{\sqrt{3}}{2} \ln x) + c)$$

$$27) \int \frac{e^{2x} dx}{\sqrt[3]{1+e^x}}$$

$$(ans.: \frac{3}{2} \cdot e^x \cdot \sqrt[3]{(1+e^x)^2} - \frac{9}{10} \sqrt[3]{(1+e^x)^5} + c)$$

$$28) \int \frac{dy}{y(2y^3+1)^2}$$

$$(ans.: \frac{1}{3} \ln(\frac{2y^3}{2y^3+1}) - \frac{2y^3}{3(2y^3+1)} + c)$$

$$29) \int \frac{x dx}{1+\sqrt{x}}$$

$$(ans.: \frac{2}{3} \sqrt{x^3} - x + 2\sqrt{x} - 2 \ln(\sqrt{x} + 1) + c)$$

$$30) \int \frac{dt}{e^t - 1}$$

$$(ans.: \ln(e^t - 1) - t + c)$$

$$\begin{array}{ll}
31) \int \frac{d\theta}{1 - \tan^2 \theta} & (\text{ans.: } \frac{1}{2}\theta + \frac{1}{4}\ln|\sec 2\theta + \tan 2\theta| + c) \\
32) \int e^x \cdot \cos 2x \, dx & (\text{ans.: } \frac{e^x}{5}\cos 2x + \frac{2}{5}e^x \sin 2x + c) \\
33) \int \frac{\cot \theta \, d\theta}{1 + \sin^2 \theta} & (\text{ans.: } \ln \frac{\sin \theta}{\sqrt{1 + \sin^2 \theta}} + c) \\
34) \int \frac{e^{4t}}{(1 + e^{2t})^{\frac{2}{3}}} dt & (\text{ans.: } \frac{3}{2}e^{2t}(1 + e^{2t})^{\frac{1}{3}} - \frac{9}{8}(1 + e^{2t})^{\frac{3}{4}} + c) \\
35) \int \frac{x^3 + x^2}{x^2 + x - 2} dx & (\text{ans.: } \frac{x^2}{2} + \frac{4}{3}\ln(x + 2) + \frac{2}{3}\ln(x - 1) + c) \\
36) \int \frac{2e^{2x} - e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx & (\text{ans.: } \frac{1}{3}(2\sqrt{3e^{2x} - 6e^x - 1} + \sqrt{3} \ln|\sqrt{3}(e^x - 1) + \sqrt{3e^{2x} - 6e^x - 1}| + c) \\
37) \int \frac{dy}{(2y + 1)\sqrt{y^2 + y}} & (\text{ans.: } \sec^{-1}(2y + 1) + c) \\
38) \int (1 - x^2)^{\frac{3}{2}} dx & (\text{ans.: } \frac{e^x}{5}\cos 2x + \frac{2}{5}e^x \sin 2x + c) \\
39) \int \frac{\tan^{-1} x}{x^2} dx & (\text{ans.: } \ln \frac{x}{\sqrt{x^2 + 1}} - \frac{\tan^{-1} x}{x^2} + c) \\
40) \int x \cdot \sin^2 x \, dx & (\text{ans.: } \frac{x^2}{4} - \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + c) \\
41) \int \frac{dt}{t^4 + 4t^2 + 3} & (\text{ans.: } \frac{1}{2}\tan^{-1} t - \frac{1}{2\sqrt{3}}\tan^{-1} \frac{t}{\sqrt{3}} + c) \\
42) \int \frac{8dx}{x^4 + 2x^3} & (\text{ans.: } \ln \frac{x}{x + 2} + \frac{2}{x} - \frac{2}{x^2} + c) \\
43) \int \frac{\cos x \, dx}{\sqrt{1 + \cos x}} & (\text{ans.: } \sqrt{2}(2 \sin \frac{x}{2} - \ln|\sec \frac{x}{2} + \tan \frac{x}{2}|) + c) \\
44) \int \frac{x \, dx}{x + \sqrt{x} + 1} & (\text{ans.: } x - 2\sqrt{x} + \frac{4}{\sqrt{3}}\tan^{-1} \frac{2\sqrt{x} + 1}{\sqrt{3}} + c) \\
45) \int \frac{dt}{\sec^2 t + \tan^2 t} & (\text{ans.: } \sqrt{2}\tan^{-1}(\sqrt{2}\tan t) - t + c)
\end{array}$$

$$\begin{array}{ll}
46) \int \frac{dx}{1 + \cos^2 x} & (\text{ans. : } \frac{1}{\sqrt{2}} \tan^{-1}(\frac{1}{\sqrt{2}} \tan x) + c) \\
47) \int \ln(\sqrt{x} + \sqrt{1+x}) dx & (\text{ans. : } x \ln(\sqrt{x} + \sqrt{1+x}) - \frac{\sqrt{x^2+x}}{2} \\
& \quad + \frac{1}{4} \ln|2x+1+2\sqrt{x^2+x}| + c) \\
48) \int x \ln(x^3 + x) dx & (\text{ans. : } \frac{x^2}{2} \ln(x^3 + x) - \frac{3}{4} x^2 + \frac{1}{2} \ln(x^2 + 1) + c) \\
49) \int \frac{\cos x dx}{\sqrt{4 - \cos^2 x}} & (\text{ans. : } \ln|\sqrt{3 + \sin^2 x} + \sin x| + c) \\
50) \int \frac{\sec^2 x dx}{\sqrt{4 - \sec^2 x}} & (\text{ans. : } \sin^{-1}(\frac{1}{\sqrt{3}} \tan x) + c) \\
51) \int \frac{dt}{t - \sqrt{1-t^2}} & (\text{ans. : } \frac{1}{2} \ln(t - \sqrt{1-t^2}) - \frac{1}{2} \sin^{-1} t + c) \\
52) \int e^{-x} \cdot \tan^{-1} e^x dx & (\text{ans. : } -e^{-x} \cdot \tan^{-1} e^x + x - \frac{1}{2} \ln(1 + e^{2x}) + c) \\
53) \int \sin^{-1} \sqrt{x} dx & (\text{ans. : } x \sin^{-1} \sqrt{x} - \frac{1}{2} \sin^{-1} \sqrt{x} + \frac{1}{2} \sqrt{x-x^2} + c) \\
54) \int \frac{\cos 2x - 1}{\cos 2x + 1} dx & (\text{ans. : } x - \tan x + c)
\end{array}$$

Chapter seven

Application of integrals

7-1- Definite integrals:

If $f(x)$ is continuous in the interval $a \leq x \leq b$ and it is integrable in the interval then the area under the curve:-

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

where $F(x)$ is any function such that $F'(x) = f(x)$ in the interval.

Some of the more useful properties of the definite integral are:-

$$1) \int_a^b c f(x) dx = c \int_a^b f(x) dx, \text{ where } c \text{ is constant.}$$

$$2) \int_a^b (f(x) \mp g(x)) dx = \int_a^b f(x) dx \mp \int_a^b g(x) dx$$

$$3) \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$4) \text{ Let } a < c < b \text{ then } \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$5) \int_a^a f(x) dx = 0$$

$$6) \text{ If } f(x) \geq 0 \text{ for } a \leq x \leq b \text{ then } \int_a^b f(x) dx \geq 0$$

$$7) \text{ If } f(x) \leq g(x) \text{ for } a \leq x \leq b \text{ then } \int_a^b f(x) dx \leq \int_a^b g(x) dx$$