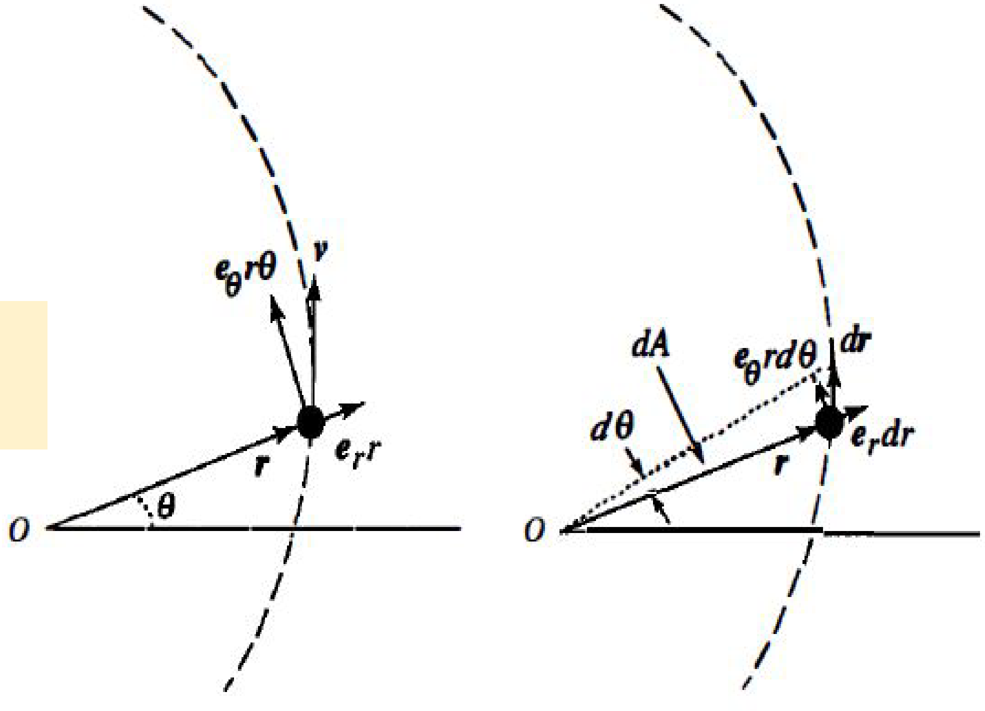
Angular Momentum and Areal Velocity of a Particle Moving in a Central Field



(a) (b)

L any particle moving in a central field of force =conserved

- we first calculate the magnitude of the angular momentum of a particle moving in a central field.
- We use polar coordinates to describe the motion
- The velocity of the particle is

$$v = e_r\dot{r} + e_\theta r\dot{\theta}$$

In the Polar coordinates(see Chapter 1)

And we have :

$$L = r \times p$$

So, the magnitude will be: $L = |r \times mv|$ → $L = |re_r \times m(e_r\dot{r} + e_\theta r\dot{\theta})|$

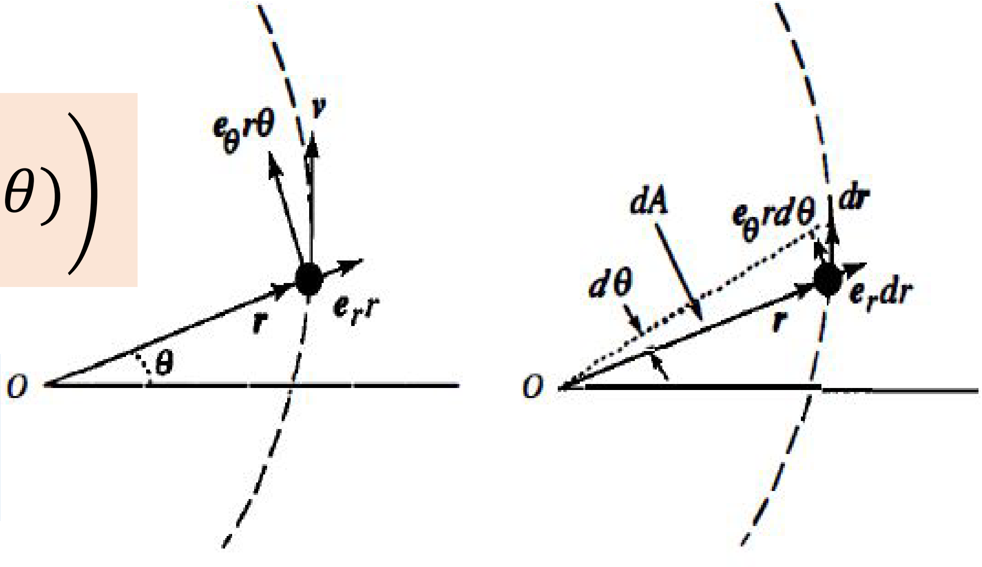
→ $L = mr^2\dot{\theta}$ =constant $as\ e_r \times e_r = 0\ and\ e_r \times e_\theta = 1$

Now, we calculate the "areal velocity," $\dot{A}$, of the particle. Figure 6.4.l(b) shows the triangular area, dA, swept out by the radius vector r as a planet moves a vector distance dr in a time dt along its trajectory relative to the origin of the central field

$$dA = \frac{1}{2}\,|r \times dr| = \frac{1}{2}\,|re_r \times (e_r dr + e_\theta r d\theta)| = \frac{1}{2}\,r\,(rd\theta)\Big)$$

$$\frac{dA}{dt} = \dot{A} = \frac{1}{2}\ r^2\dot{\theta} = \frac{L}{2m}$$

$$\frac{dA}{dt} = \dot{A} = \frac{L}{2m} = constant$$



(a) (b)

Thus, the areal velocity, A, of a particle moving in a central field is directly proportional to its angular momentum and, therefore, is also a constant of the motion, exactly as Kepler discovered for planets moving in the central gravitational field of the Sun.

Example (1)

Let a particle be subject to an attractive central force of the from (r) , where r is the distance between the particle and the centre of the force. Find $f\ (r)$ if all circular orbits are to have identical areal velocities, $\dot{A}$.

Solution:

Because the orbits are circular, the acceleration,r, has no transverse component and is entirely in the radial direction. In polar coordinates, the acceleration is given by:

$$a = \ddot{r} - r\dot{\theta}^2$$

Because the orbits are circular, the acceleration, $i.e. \ddot{r} = 0$,

Thus,

$$ma_r = -mr\dot{\theta}^2 = f(r) \qquad \times \left(\frac{r^3}{r^3}\right)$$

$$f(r) = -\frac{mr^4\dot{\theta}^2}{r^3} = \frac{L^2}{mr^3} = f(r) \qquad , As\ L = mr\ \dot{\theta}$$

$$f(r) = -\frac{4m\dot{A}^2}{r^3} = f(r) \qquad , As\ \dot{A} = \frac{L}{2m}$$

6.5 Kepler's First Law: The law of Ellipses :

To prove Kepler's first law, we develop a general differential equation for the orbit of a particle in any central, isotropic field of force. Then we solve the orbital equation for the specific case of an inverse-square law of force.

The equation of motion in polar coordinates is $m\ddot{r} = f(r)e_r$

Where $f(r)$ is the central, isotropic force that acts on the particle of mass m.

acceleration vector in polar coordinates

$$a = \ddot{r} = (\ddot{r} - r\dot{\theta}^2)e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta$$

So,

$$m(\ddot{r} - r\dot{\theta}^2)e_r = f(r)$$

$$\mathrm{m}(r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta = 0$$

No component toward θ direction

$$\frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

Or

$$r^2\dot{\theta} = constant = l$$

Where l is the angular momentum per unit mass:

$$\mathbf{m}\left(\boldsymbol{r\ddot{\theta}} + \mathbf{2}\boldsymbol{\dot{r}\dot{\theta}}\right)\boldsymbol{e_\theta} = \mathbf{0}$$

$$l = \frac{L}{m} = |r \times v|$$

Given a certain radial force function f (r), we could, in theory, solve the pair of differential equations (Equations 6.10a and b) to obtain r and θ as functions of t. Often one is interested only in the path in space (the orbit) without regard to the time t. **To find the equation of the orbit, we use the variable u defined by**

$$r = \frac{1}{u} \text{ or } u = \frac{1}{r}$$

$$\text{And } l = r^2\dot{\theta} = \frac{1}{u^2}\,\dot{\theta}$$

$$dr = \dot{r} = \frac{-1}{u^2}\dot{u} = \frac{-1}{u^2}\;\frac{du}{d\theta}\frac{d\theta}{du} = \frac{-1}{u^2}\;\dot{\theta}\frac{du}{d\theta} = -\;l\;\frac{du}{d\theta}$$

$$dr = \dot{r} = \frac{-1}{u^2}\dot{u} = \frac{-1}{u^2} \quad \frac{du}{d\theta}\frac{d\theta}{du} = \frac{-1}{u^2} \quad \dot{\theta}\frac{du}{d\theta} = -\ l\ \frac{du}{d\theta}$$

As we employed the fact $l = \dot{\theta}u^2$ So the above equation can be written as:

$$\dot{r} = -\ l\ \frac{du}{d\theta} \xrightarrow{\text{H.W}} \ddot{r} = -\ l^2u^2\ \frac{d^2u}{d\theta^2}$$

Substituting the values found for r, $\dot{\theta}$, and $\ddot{r}$ into Equation 6.10a, we obtain

$$a = \ddot{r} = \left(\ddot{r} - r\dot{\theta}^2\right)e_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)e_\theta$$

$$m\left(\ddot{r} - r\dot{\theta}^2\right)e_r = f(r) \qquad \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)e_\theta = 0$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2}f\left(u^{-1}\right)$$

→ Differential equation of the orbit of a particle moving under a central force.

Example (2):

A particle in a central field moves in the spiral orbit $r = c\theta^2$ Determine the force function.

Solution:

We have $u = \frac{1}{r} = \frac{1}{c\theta^2}$ and $\theta = \frac{1}{\sqrt{cu}}$ → $\frac{du}{d\theta} = -\frac{2}{c}\frac{1}{\theta^3}$

$$\frac{d^2u}{d\theta^2} = -\frac{6}{c}\frac{1}{\theta^4} = 6\,cu^2$$

Now, eq. 6.17 will applied

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2} f(u^{-1})$$

$$6\,cu^2 + u = -\frac{1}{ml^2u^2} f(u^{-1})$$

$$f(u^{-1}) = -ml^2(6cu^2 + u^3)$$

$$f(r) = -ml^2\left(\frac{6c}{r^4} + \frac{1}{r^3}\right) \quad \text{as } u = 1/r$$

Thus, the force is a combination of an inverse cube and inverse-fourth power law