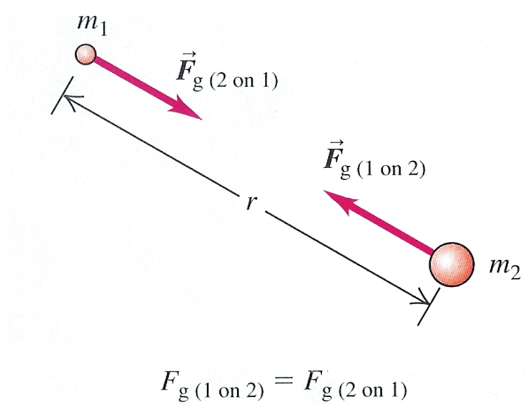
6.1. Newton's Law of Gravitation:

- During his study of the motions of the planets and of the moon, Newton discovered the fundamental charter of the gravitational attraction between any two bodies. In 1687, Newton published the law of gravitation which may be stated as follows:

*Every particle of matter in the universe attracts every other particle with **force** (F_g) that is directly proportional to the product of the masses (m_1, m_2) of the particles and **inversely proportional to the square of the distance between them** (r).*



Or;

$$F_g = \frac{G m_1 m_2}{r^2} \qquad (131)$$

G is a fundamental physical constant called the ***gravitational constant***. The numerical value of G (in SI units) is

$$G = 6.67\times10^{-11}\ N.m^2/kg^2$$

- Equation (131) tells us that if the distance r is doubled, the force is only one-fourth as great, and so on. Although many of the stars in the night sky are more massive than the sun, they are so far away that their gravitational force on earth is negligible.

- Gravitational forces always act along the line joining the two particles, which form an action-reaction pair. Even when the masses of the particles are different, the two interaction forces have equal magnitude.

- Gravitational forces combine vectorially. If each of two masses exerts a force on a third, the total force on the third mass is the vector sum of the individual forces of the first two. This property is often called *superposition of forces*.

- The earth's gravitational force on a body of mass m at any point outside the earth is given by ; $F_g = Gm_Em/r^2$, where m_E is the mass of the earth and r is the distance of the body from the earth's center. Therefore, we can express the ***gravitational potential energy*** (U) in more general form as;

$$U_g = -\frac{Gm_Em}{r} \qquad (132)$$

6.2. Kepler's laws and the motion of planets:

- One of the great intellectual events of the 16th and 17th centuries was the threefold realization;

1- that the earth is also a planet,

2- that all planets orbit the sun,

3- and that the apparent motions of the planets as seen from the earth can be used to determine the orbits of the planets precisely.

The first and second of these ideas were published by *Nicolaus Copernicus* in 1543. The determination of planetary orbits was carried out between 1601 and 1619 by the German astronomer and mathematician *Johannes Kepler*.

- By trial and error, *Kepler* discovered **three observed laws** that accurately described the motions of the planets. Two hundred years later, Newton discovered that each of Kepler's laws can be derived using Newton's laws of motion and the law of gravitation.

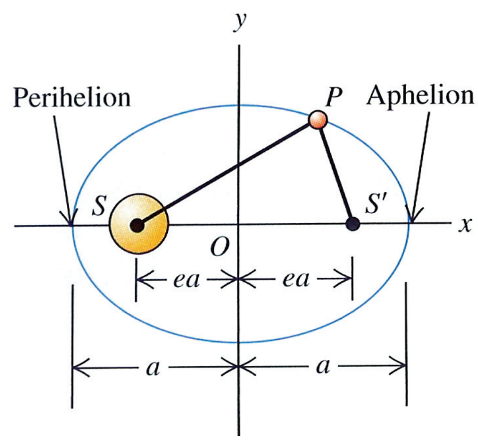
First law: Law of Ellipses.

Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.

The following Figure shows the geometry of the ellipse with its main properties;

* The longest dimension $2a$ is the major axis, with half-length a known as the ***semi-major axis***.

* S and S' are the *foci* (plural of focus). The sun is at S, and the planet is at P.



* The sum of the distances from S to P and from S' to P is the same for all points on the curve.

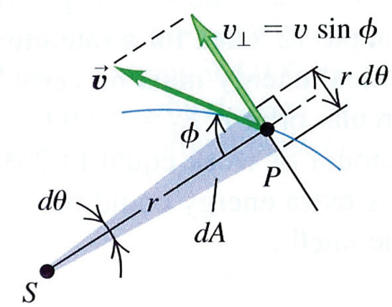
* The distance of each focus from the center of the ellipse is ea, where e is a dimensionless number between 0 and 1 called the ***eccentricity***. If e = 0, the ellipse is a circle. The actual orbits of the planets are somewhat circular; their eccentricities range from 0.007 for Venus to 0.248 for Pluto. The earth's orbit has e = 0.017.

* The point in the planet's orbit closest to the sun is the ***perihelion***, and the point most distant from the sun is the ***aphelion***.

Scecond law: Law of Equal Areas

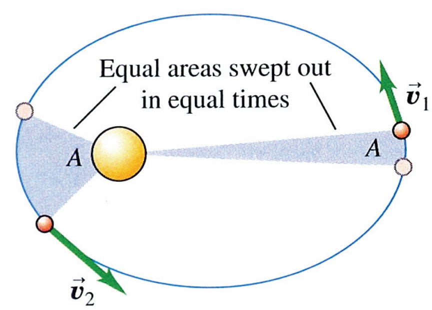
A line from the sun to a given planet sweeps out equal areas in equal times.

In a small time interval dt, the line from the sun S to the planet P turns through an angle $d\theta$. The area swept out is the $dA = \frac{1}{2} r^2 \, d\theta$ .The rate at which area is swept out, dA/dt, is called the *sector velocity* :



$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

The real meaning of Kepler's second law is that the ***sector velocity has the same value at all points in the orbit***. When the planet is close to the sun, r is small and $d\theta/dt$ is large; when the planet is far from the sun, r is large and $d\theta/dt$ is small.



Third law: Harmonic Law

The square of the period of a planet is directly proportional to the cube of the semi-major axis of the plant's orbit.

This law can be expressed as;

$$T^2 = ka^3$$

If the distance measured in **astronomical units** (**1AU**=1.5 $\times 10^8$ km) and periods are measured in Earth **years** then: $k=1$.

Planet	$\tau(yr)$	$\tau^2(yr^2)$	a(AU)	a^3(AU^3)	ϵ
Mercury	0.241	**0.0581**	0.387	**0.0580**	0.206
Venus	0.615	**0.378**	0.723	**0.378**	0.007
Earth	1.000	**1.000**	1.000	**1.000**	0.017
Mars	1.881	**3.538**	1.524	**3.540**	0.093
Jupiter	11.86	**140.7**	5.203	**140.8**	0.048
Saturn	29.46	**867.9**	9.539	**868.0**	0.056
Uranus	84.01	**7058.**	19.18	**7056.**	0.047
Neptune	164.8	**27160.**	30.06	**27160.**	0.009
Pluto	247.7	**61360.**	39.440	**61350.**	0.249

EXAMPLE (6.6.1):

Find the period of a comet whose semi-major axis is 4 AU.

Solution:

With T measured in years and a in astronomical units, we have

$$T^2 = a^3 = (4)^3 = 64 \text{ yrs}^2$$

$$T = 8 \text{ yrs}$$

However, using Newton's laws of motion and the inverse-square law of gravity, one can found (see problem 6.5) that the constant k in **SI unit** is equals to;

$$k = \frac{4\pi^2}{Gm_s}$$

Where m_s is the sun's mass.

Kepler's 3rd law can be then rewritten as

$$T = 2\pi \frac{a^{3/2}}{\sqrt{Gm_s}} \qquad (133)$$

Note:

> The *period* does not depend on the eccentricity e. I.e. an asteroid in an elliptical orbit with semi-major axis a will have the same orbital period as a planet in a circular orbit of radius a. The key difference is that the asteroid moves at different speeds at different points in its elliptical orbit, while the planet's speed is constant around its circular orbit.

Example : Comet Halley

Comet Halley moves in an elongated elliptical orbit around the sun. At perihelion, the comet is 8.75×10^7 km from the sun; at aphelion it is 5.26×10^9 km from the sun. Find the semi- major axis, eccentricity, and period of the orbit.

Solution:

- The length of the major axis is;

$$2a = 8.75 \times 10^7 + 5.26 \times 10^9$$

So; $$a = 2.67 \times 10^9 \text{ km}$$

- Since the comet-sun distance at perihelion is given by

$$a - ea = a\,(1- e) = 8.75 \times 10^7$$

Then; $$e = 0.967$$

- From Eq. (133),

$$T = 2\pi \frac{(2.67\times10^{12})^{3/2}}{\sqrt{(6.67\times10^{-11})(1.99\times10^{30})}} = 2.38\times10^9 s = 75.5 \text{ years}$$