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الرياضيات التكاملية

محاضرة 4

The substitution rule, Integration by parts

التكاملات بالاستبدال والتجزئة

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The substitution rule

If u = g(x) is a differentiable function whose range is an interval I and f is continuous on I , then for indefinite integral

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

And for a definite integral

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

4. Integration by parts

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

We can rearrange this equation as

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

The general form of the rule becomes

$$\int u\,dv = uv - \int v\,du$$

Examples

1-The Method of Substitution

1- evaluate the following integrals by using the Substitution method:

a- $\int e^{5-2x}\,dx$ Let $u = 5-2x$

b- $\int \cos(ax+b)\,dx$ Let $u = ax+b$

c- $\int e^{2x}\sin(e^{2x})\,dx$ Let $u = e^{2x}$

d- $\int \frac{x\,dx}{(4x^2+1)^5}$ Let $u = 4x^2+1$

e- $\int \frac{t}{\sqrt{4-t^4}}\,dt$ Let $u = t^2$

f- $\int \frac{dx}{e^x+1}$

g- $\int \frac{dx}{e^x+e^{-x}}$

h- $\int \frac{x+1}{\sqrt{1-x^2}}\,dx$ Let $u = 1-x^2$

i- $\int \frac{dx}{x^2+6x+13}$

j- $\int_0^4 x^3(x^2+1)^{-1/2}\,dx$ Let $u = x^2+1$

k- $\int_0^2 \frac{x}{x^2+16}\,dx$ Let $u = x^2+16$

$\int_0^2 \frac{x\,dx}{x^4+16}$ Let $u = x^2$

Solution

a- $\int e^{5-2x}\,dx$ Let $u = 5 - 2x$

$du = -2\,dx$

$= -\frac{1}{2}\int e^{u}\,du = -\frac{1}{2}e^{u} + C = -\frac{1}{2}e^{5-2x} + C.$

b- $\int \cos(ax+b)\,dx$ Let $u = ax + b$

$du = a\,dx$

$= \frac{1}{a}\int \cos u\,du = \frac{1}{a}\sin u + C$

$= \frac{1}{a}\sin(ax+b) + C.$

c- $\int e^{2x}\sin(e^{2x})\,dx$ Let $u = e^{2x}$

$du = 2e^{2x}\,dx$

$= \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C$

$= -\frac{1}{2}\cos(e^{2x}) + C.$

d- $\int \frac{x\,dx}{(4x^2+1)^5}$ Let $u = 4x^2 + 1$

$du = 8x\,dx$

$= \frac{1}{8}\int u^{-5}\,du = -\frac{1}{32}u^{-4} + C = \frac{-1}{32(4x^2+1)^4} + C$

e- $\int \frac{t}{\sqrt{4-t^4}}\,dt$ Let $u = t^2$

$du = 2t\,dt$

$= \frac{1}{2}\int \frac{du}{\sqrt{4-u^2}}$

$= \frac{1}{2}\sin^{-1}\frac{u}{2} + C = \frac{1}{2}\sin^{-1}\left(\frac{t^2}{2}\right) + C..$

f- $\int \frac{dx}{e^x+1} = \int \frac{e^{-x}\,dx}{1+e^{-x}}$ Let $u = 1 + e^{-x}$
$du = -e^{-x}\,dx$

$$= -\int \frac{du}{u} = -\ln|u| + C = -\ln(1+e^{-x}) + C.$$

g- $\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x\,dx}{e^{2x}+1}$ Let $u = e^x$
$du = e^x\,dx$

$$= \int \frac{du}{u^2+1} = \tan^{-1} u + C$$
$$= \tan^{-1} e^x + C.$$

h- $\int \frac{x+1}{\sqrt{1-x^2}}\,dx$

$= \int \frac{x\,dx}{\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1-x^2}}$ Let $u = 1 - x^2$
$du = -2x\,dx$

in the first integral only

$$= -\frac{1}{2}\int \frac{du}{\sqrt{u}} + \sin^{-1} x = -\sqrt{u} + \sin^{-1} x + C$$
$$= -\sqrt{1-x^2} + \sin^{-1} x + C.$$

i- $\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{(x+3)^2+4}$ Let $u = x + 3$
$du = dx$

$$= \int \frac{du}{u^2+4} = \frac{1}{2}\tan^{-1}\frac{u}{2} + C$$
$$= \frac{1}{2}\tan^{-1}\frac{x+3}{2} + C.$$

j- $\int_0^4 x^3(x^2+1)^{-1/2}\,dx$ Let $u = x^2 + 1$, $x^2 = u - 1$

$$= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right)\Bigg|_1^{17}$$
$$= \frac{17\sqrt{17}-1}{3} - (\sqrt{17}-1) = \frac{14\sqrt{17}}{3} + \frac{2}{3}.$$

k- $\int_0^2 \frac{x}{x^2+16}\,dx$ Let $u = x^2 + 16$

$du = 2x\,dx$

$= \frac{1}{2}\int_{16}^{20} \frac{du}{u} = \frac{1}{2}\ln u\Big|_{16}^{20}$

$= \frac{1}{2}(\ln 20 - \ln 16) = \frac{1}{2}\ln\left(\frac{5}{4}\right)$

l- $\int_0^2 \frac{x\,dx}{x^4+16}$ Let $u = x^2$

$du = 2x\,dx$

$= \frac{1}{2}\int_0^4 \frac{du}{u^2+16} = \frac{1}{8}\tan^{-1}\frac{u}{4}\Big|_0^4 = \frac{\pi}{32}$

Integration by parts

Evaluate the following integrals

a- $\int x \sin x\,dx$

b- $\int x \cos x\,dx$

c- $\int (x+3)e^{2x}\,dx$

d- $\int (x^2 - 2x)e^{kx}\,dx$

e- $\int \tan^{-1} x\,dx$

f- $\int \ln u\,du$

g- $\int \cos^{-1} x\,dx$

h- $\frac{1}{2}\int_0^{\pi} x \sin x\,dx$

i- $\frac{1}{2}\int_0^{\pi} x^2 \sin x\,dx$

Solution

$$\int x \sin x \, dx$$

$$u = x \qquad dv = \sin x \, dx$$

$$du = dx \qquad v = -\cos x$$

$$\int x \sin x \, dx = \int \overbrace{x}^{u} \, \overbrace{\sin x \, dx}^{dv} = \overbrace{x}^{u} \, \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v} \, \overbrace{dx}^{du}$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

$$\int x \cos x \, dx$$

$$U = x \qquad dV = \cos x \, dx$$

$$dU = dx \qquad V = \sin x$$

$$= x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + C.$$

$$\int (x+3)e^{2x} \, dx$$

$$U = x + 3 \qquad dV = e^{2x} \, dx$$

$$dU = dx \qquad V = \tfrac{1}{2} e^{2x}$$

$$= \frac{1}{2}(x+3)e^{2x} - \frac{1}{2} \int e^{2x} \, dx$$

$$= \frac{1}{2}(x+3)e^{2x} - \frac{1}{4} e^{2x} + C.$$

$$\int (x^2 - 2x)e^{kx}\, dx$$

$$\begin{aligned} U &= x^2 - 2x & dV &= e^{kx} \\ dU &= (2x - 2)\, dx & V &= \frac{1}{k}e^{kx} \end{aligned}$$

$$= \frac{1}{k}(x^2 - 2x)e^{kx} - \frac{1}{k}\int (2x - 2)e^{kx}\, dx$$

$$\begin{aligned} U &= x - 1 & dV &= e^{kx}\, dx \\ dU &= dx & V &= \frac{1}{k}e^{kx} \end{aligned}$$

$$= \frac{1}{k}(x^2 - 2x)e^{kx} - \frac{2}{k}\left[\frac{1}{k}(x - 1)e^{kx} - \frac{1}{k}\int e^{kx}\, dx\right]$$

$$= \frac{1}{k}(x^2 - 2x)e^{kx} - \frac{2}{k^2}(x - 1)e^{kx} + \frac{2}{k^3}e^{kx} + C.$$

$$\int \tan^{-1} x\, dx$$

$$\begin{aligned} U &= \tan^{-1} x & dV &= dx \\ dU &= \frac{dx}{} & V &= x \end{aligned}$$

$$= x\tan^{-1} x - \int \frac{x\, dx}{1 + x^2}$$

$$= x\tan^{-1} x - \frac{1}{2}\ln(1 + x^2) + C.$$

$$\int \ln u\, du$$

$$\begin{aligned} U &= \ln u & dV &= du \\ dU &= \frac{du}{u} & V &= u \end{aligned}$$

$$= u\ln u - \int du = u\ln u - u + C$$

$$= (\ln x)(\ln(\ln x)) - \ln x + C.$$

$$\int \cos^{-1} x \, dx$$

$$U = \cos^{-1} x \qquad dV = dx$$

$$dU = -\frac{dx}{\sqrt{1-x^2}} \qquad V = x$$

$$= x \cos^{-1} x + \int \frac{x \, dx}{\sqrt{1-x^2}}$$

$$= x \cos^{-1} x - \sqrt{1-x^2} + C.$$

$$\frac{1}{2} \int_0^{\pi} x \sin x \, dx$$

$$U = x \qquad dV = \sin x \, dx$$

$$dU = dx \qquad V = -\cos x$$

$$= \frac{1}{2} \left[-x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx \right]$$

$$= \frac{\pi}{2} = 1.571.$$

$$\frac{1}{2} \int_0^{\pi} x^2 \sin x \, dx$$

$$U = x^2 \qquad dV = \sin x \, dx$$

$$dU = 2x \, dx \qquad V = -\cos x$$

$$= \frac{1}{2} \left[-x^2 \cos x \Big|_0^{\pi} + 2 \int_0^{\pi} x \cos x \, dx \right]$$

$$U = x \qquad dV = \cos x \, dx$$

$$= \frac{1}{2} \left[\pi^2 + 2 \left(x \sin x \Big|_0^{\pi} - \int_0^{\pi} \sin x \, dx \right) \right]$$

$$= \frac{1}{2}(\pi^2 - 4).$$