1. Function Composition

The composition of *f(x)* and *g(x)* is

$$(f \circ g)(x) = f(g(x))$$

In other words, compositions are evaluated by plugging the second function listed into the first function listed. Note as well that order is important here. Interchanging the order will more often than not result in a different answer.

Example.1

Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$ find each of the following.

(a) $(f \circ g)(5)$

(b) $(f \circ g)(x)$

(c) $(g \circ f)(x)$

(d) $(g \circ g)(x)$

Solution:

(a) $(f \circ g)(5)$

In this case we've got a number instead of an x but it works in exactly the same way.

$$\begin{aligned}(f \circ g)(5) &= f(g(5)) \\ &= f(-99) = 29512\end{aligned}$$

(b) $(f \circ g)(x)$

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$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(1-20x) \\ &= 3(1-20x)^2 - (1-20x) + 10 \\ &= 3\left(1-40x+400x^2\right) - 1 + 20x + 10 \\ &= 1200x^2 - 100x + 12\end{aligned}$$

Compare this answer to the next part and notice that answers are NOT the same. The order in which the functions are listed is important!

(c) $(g \circ f)(x)$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g\left(3x^2 - x + 10\right) \\ &= 1 - 20\left(3x^2 - x + 10\right) \\ &= -60x^2 + 20x - 199\end{aligned}$$

And just to make the point one more time. This answer is different from the previous part. Order is important in composition.

(d) $(g \circ g)(x)$

In this case do not get excited about the fact that it's the same function. Composition still works the same way.

$$\begin{aligned}(g \circ g)(x) &= g(g(x)) \\ &= g(1-20x) \\ &= 1 - 20(1-20x) \\ &= 400x - 19\end{aligned}$$

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Example.2

Given $f(x) = 3x - 2$ and $g(x) = \frac{1}{3}x + \frac{2}{3}$ find each of the following.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

Solution:

(a) $(f \circ g)(x)$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{3}x + \frac{2}{3}\right) \\ &= 3\left(\frac{1}{3}x + \frac{2}{3}\right) - 2 \\ &= x + 2 - 2 \\ &= x\end{aligned}$$

(b) $(g \circ f)(x)$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x - 2) \\ &= \frac{1}{3}(3x - 2) + \frac{2}{3} \\ &= x - \frac{2}{3} + \frac{2}{3} \\ &= x\end{aligned}$$

2. Inverse Functions

The process for finding the inverse of a function is a fairly simple one although there are a couple of steps that can on occasion be somewhat messy. Here is the process.

Given the function $f(x)$ we want to find the inverse function, $f^{-1}(x)$.

1. First, replace $f(x)$ with y. This is done to make the rest of the process easier.
2. Replace every x with a y and replace every y with an x.
3. Solve the equation from Step 2 for y. This is the step where mistakes are most often made so be careful with this step.
4. Replace y with $f^{-1}(x)$. In other words, we've managed to find the inverse at this point!
5. Verify your work by checking that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true. This work can sometimes be messy making it easy to make mistakes so again be careful.

Example.3

Given $f(x) = 3x - 2$ find $f^{-1}(x)$.

Solution:

So, let's get started. We'll first replace $f(x)$ with y.

$$y = 3x - 2$$

Next, replace all x's with y and all y's with x.

$$x = 3y - 2$$

Now, solve for y.

$$
\begin{aligned}
x+2 &= 3y \\
\frac{1}{3}(x+2) &= y \\
\frac{x}{3}+\frac{2}{3} &= y
\end{aligned}
$$

Finally replace y with $f^{-1}(x)$.

$$f^{-1}(x)=\frac{x}{3}+\frac{2}{3}$$

Now, we need to verify the results. We already took care of this in the previous section, however, we really should follow the process so we'll do that here. It doesn't matter which of the two that we check we just need to check one of them. This time we'll check that $\left(f\circ f^{-1}\right)(x)=x$ is true.

$$
\begin{aligned}
\left(f\circ f^{-1}\right)(x) &= f\left[f^{-1}(x)\right] \\
&= f\left[\frac{x}{3}+\frac{2}{3}\right] \\
&= 3\left(\frac{x}{3}+\frac{2}{3}\right)-2 \\
&= x+2-2 \\
&= x
\end{aligned}
$$

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Example.4

Given $g(x) = \sqrt{x-3}$ find $g^{-1}(x)$.

Solution:

$$y = \sqrt{x-3} \qquad \Rightarrow \qquad x = \sqrt{y-3}$$

Now, to solve for y we will need to first square both sides and then proceed as normal.

$$\begin{aligned} x &= \sqrt{y-3} \\ x^2 &= y-3 \\ x^2+3 &= y \end{aligned}$$

This inverse is then,

$$g^{-1}(x) = x^2 + 3$$

Finally let's verify and this time we'll use the other one just so we can say that we've gotten both down somewhere in an example.

$$\begin{aligned} \left(g^{-1} \circ g\right)(x) &= g^{-1}\left[g(x)\right] \\ &= g^{-1}\left(\sqrt{x-3}\right) \\ &= \left(\sqrt{x-3}\right)^2 + 3 \\ &= x - 3 + 3 \\ &= x \end{aligned}$$

So, we did the work correctly and we do indeed have the inverse.

H.W

Given $h(x) = \dfrac{x+4}{2x-5}$ find $h^{-1}(x)$.

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3. The limit

We say that the limit of $f(x)$ is L as x approaches a and write this as

$$\lim_{x \to a} f(x) = L$$

provided we can make $f(x)$ as close to L as we want for all x sufficiently close to a, from both sides, without actually letting x be a.

First, we will assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and that c is any constant. Then,

1. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
2. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \quad \lim_{x \to a} g(x)$
4. $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$
5. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$, where n is any real number
6. $\lim_{x \to a} \left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x \to a} f(x)}$
7. $\lim_{x \to a} c = c, \quad c$ is any real number
   In other words, the limit of a constant is just the constant. You should be able to convince yourself of this by drawing the graph of $f(x) = c$.
8. $\lim_{x \to a} x = a$
   As with the last one you should be able to convince yourself of this by drawing the graph of $f(x) = x$.
9. $\lim_{x \to a} x^n = a^n$
   This is really just a special case of property **5** using $f(x) = x$.

Example. 5:

Compute the value of the following limit.

$$\lim_{x \to -2} (3x^2 + 5x - 9)$$

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Solution:

$$\lim_{x \to -2} \left(3x^2 + 5x - 9\right) = \lim_{x \to -2} 3x^2 + \lim_{x \to -2} 5x - \lim_{x \to -2} 9$$
$$= 3 \lim_{x \to -2} x^2 + 5 \lim_{x \to -2} x - \lim_{x \to -2} 9$$

We can now use properties **7** through **9** to actually compute the limit.

$$\lim_{x \to -2} \left(3x^2 + 5x - 9\right) = 3 \lim_{x \to -2} x^2 + 5 \lim_{x \to -2} x - \lim_{x \to -2} 9$$
$$= 3(-2)^2 + 5(-2) - 9$$
$$= -7$$

Example. 7:

Evaluate the following limit.

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1}$$

Solution:

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} = \frac{\lim_{z \to 1} 6 - 3z + 10z^2}{\lim_{z \to 1} -2z^4 + 7z^3 + 1}$$

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} = \frac{6 - 3(1) + 10(1)^2}{-2(1)^4 + 7(1)^3 + 1}$$
$$= \frac{13}{6}$$

H.W

Evaluate the following limit.

$$\lim_{x \to 3} \left(-\sqrt[5]{x} + \frac{\mathbf{e}^x}{1 + \ln(x)} + \sin(x)\cos(x)\right)$$

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