1. Derivatives of Trig Functions

Before we actually get into the derivatives of the trig functions, we need to give a couple of limits that will show up in the derivation of two of the derivatives.

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

Example 1:

Evaluate each of the following limits.

(a) $\lim_{\theta \to 0} \frac{\sin\theta}{6\theta}$

(b) $\lim_{x \to 0} \frac{\sin(6x)}{x}$

(c) $\lim_{x \to 0} \frac{x}{\sin(7x)}$

(d) $\lim_{t \to 0} \frac{\sin(3t)}{\sin(8t)}$

(e) $\lim_{x \to 4} \frac{\sin(x-4)}{x-4}$

(f) $\lim_{z \to 0} \frac{\cos(2z)-1}{z}$

Solution

(a) $\lim_{\theta \to 0} \frac{\sin\theta}{6\theta}$

$$\lim_{\theta \to 0} \frac{\sin\theta}{6\theta} = \frac{1}{6}\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \frac{1}{6}(1) = \frac{1}{6}$$

(b) $\lim\limits_{x\to 0} \frac{\sin(6x)}{x}$

$$\lim_{x\to 0} \frac{\sin(6x)}{x} = \lim_{x\to 0} \frac{6\sin(6x)}{6x} = 6\lim_{x\to 0} \frac{\sin(6x)}{6x}$$
$$= 6(1)$$
$$= 6$$

(c) $\lim\limits_{x\to 0} \frac{x}{\sin(7x)}$

$$\frac{x}{\sin(7x)} = \frac{1}{\frac{\sin(7x)}{x}}$$
$$= \frac{\lim\limits_{x\to 0} 1}{\lim\limits_{x\to 0} \frac{\sin(7x)}{x}}$$
$$= \frac{1}{\lim\limits_{x\to 0} \frac{\sin(7x)}{x}}$$

$$\lim_{x\to 0} \frac{x}{\sin(7x)} = \frac{1}{\lim\limits_{x\to 0} \frac{7\sin(7x)}{7x}}$$
$$= \frac{1}{7\lim\limits_{x\to 0} \frac{\sin(7x)}{7x}}$$
$$= \frac{1}{(7)(1)}$$
$$= \frac{1}{7}$$

(d) $\lim\limits_{t\to 0} \frac{\sin(3t)}{\sin(8t)}$

$$\lim_{t\to 0} \frac{\sin(3t)}{\sin(8t)} = \lim_{t\to 0} \frac{\sin(3t)}{1} \frac{1}{\sin(8t)}$$

$$\lim_{t\to 0} \frac{\sin(3t)}{\sin(8t)} = \lim_{t\to 0} \frac{\sin(3t)}{1} \frac{1}{\sin(8t)} \frac{t}{t}$$
$$= \lim_{t\to 0} \frac{\sin(3t)}{t} \frac{t}{\sin(8t)}$$

$$= \left(\lim_{t\to 0} \frac{\sin(3t)}{t}\right)\left(\lim_{t\to 0} \frac{t}{\sin(8t)}\right)$$

$$\begin{aligned}
\lim_{t\to 0} \frac{\sin(3t)}{\sin(8t)} &= \left(\lim_{t\to 0} \frac{3\sin(3t)}{3t}\right)\left(\lim_{t\to 0} \frac{8t}{8\sin(8t)}\right) \\
&= \left(3\lim_{t\to 0} \frac{\sin(3t)}{3t}\right)\left(\frac{1}{8}\lim_{t\to 0} \frac{8t}{\sin(8t)}\right) \\
&= (3)\left(\frac{1}{8}\right) \\
&= \frac{3}{8}
\end{aligned}$$

(e) $\lim_{x\to 4} \frac{\sin(x-4)}{x-4}$

$$\lim_{x\to 4} \frac{\sin(x-4)}{x-4} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

(f) $\lim_{z\to 0} \frac{\cos(2z)-1}{z}$

$$\begin{aligned}
\lim_{z\to 0} \frac{\cos(2z)-1}{z} &= \lim_{z\to 0} \frac{2(\cos(2z)-1)}{2z} \\
&= 2\lim_{z\to 0} \frac{\cos(2z)-1}{2z} \\
&= 2(0) \\
&0
\end{aligned}$$

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2. Derivatives of the six trig functions

$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x) \qquad \frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$$

$$\frac{d}{dx}\left(\tan(x)\right) = \sec^2(x) \qquad \frac{d}{dx}\left(\cot(x)\right) = -\csc^2(x)$$

$$\frac{d}{dx}\left(\sec(x)\right) = \sec(x)\tan(x) \qquad \frac{d}{dx}\left(\csc(x)\right) = -\csc(x)\cot(x)$$

Example 2:

Differentiate each of the following functions.

(a) $g(x) = 3\sec(x) - 10\cot(x)$

(b) $h(w) = 3w^{-4} - w^2\tan(w)$

(c) $y = 5\sin(x)\cos(x) + 4\csc(x)$

(d) $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$

Solution

(a) $g(x) = 3\sec(x) - 10\cot(x)$

$$\begin{aligned} g'(x) &= 3\sec(x)\tan(x) - 10\left(-\csc^2(x)\right) \\ &= 3\sec(x)\tan(x) + 10\csc^2(x) \end{aligned}$$

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(b) $h(w) = 3w^{-4} - w^2 \tan(w)$

$$\begin{aligned} h'(w) &= -12w^{-5} - \left(2w\tan(w) + w^2\sec^2(w)\right) \\ &= -12w^{-5} - 2w\tan(w) - w^2\sec^2(w) \end{aligned}$$

$$h'(w) = -12w^{-5} - 2w\tan(w) - w^2\sec^2(w)$$

(c) $y = 5\sin(x)\cos(x) + 4\csc(x)$

$$\begin{aligned} y' &= 5\cos(x)\cos(x) + 5\sin(x)(-\sin(x)) - 4\csc(x)\cot(x) \\ &= 5\cos^2(x) - 5\sin^2(x) - 4\csc(x)\cot(x) \end{aligned}$$

(d) $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$

$$\begin{aligned} P'(t) &= \frac{\cos(t)(3 - 2\cos(t)) - \sin(t)(2\sin(t))}{(3 - 2\cos(t))^2} \\ &= \frac{3\cos(t) - 2\cos^2(t) - 2\sin^2(t)}{(3 - 2\cos(t))^2} \end{aligned}$$

$$\begin{aligned} P'(t) &= \frac{3\cos(t) - 2\left(\cos^2(t) + \sin^2(t)\right)}{(3 - 2\cos(t))^2} \\ &= \frac{3\cos(t) - 2}{(3 - 2\cos(t))^2} \end{aligned}$$