Al-Mustaqbal University
Department of Electrical Engineering Technique
Class 1st
Subject: Differential mathematics
Lecturer: Dr. Hasan Muwafaq Gheni

1. The Definition of the Derivative

The derivative of $f(x)$ with respect to x is the function $f'(x)$ and is defined as,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 1:

Find the derivative of the following function using the definition of the derivative.

$$f(x) = 2x^2 - 16x + 35$$

Solution:

First plug the function into the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2(x+h)^2 - 16(x+h) + 35 - (2x^2 - 16x + 35)}{h} \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 16x - 16h + 35 - 2x^2 + 16x - 35}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2 - 16h}{h} \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{h(4x + 2h - 16)}{h} \\ &= \lim_{h \to 0} 4x + 2h - 16 \\ &= 4x - 16 \end{aligned}$$

So, the derivative is,

$$f'(x) = 4x - 16$$

Email: Hasan.muwafaq.gheni@uomus.edu.iq

Example 2:

Find the derivative of the following function using the definition of the derivative.

$$g(t) = \frac{t}{t+1}$$

Solution:

$$\begin{aligned} g'(t) &= \lim_{h\to 0} \frac{g(t+h) - g(t)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{t+h}{t+h+1} - \frac{t}{t+1}\right) \end{aligned}$$

As with the first problem we can't just plug in $h = 0$. So, we will need to simplify things a little. In this case we will need to combine the two terms in the numerator into a single rational expression as follows.

$$\begin{aligned} g'(t) &= \lim_{h\to 0} \frac{1}{h}\left(\frac{(t+h)(t+1) - t(t+h+1)}{(t+h+1)(t+1)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{t^2 + t + th + h - (t^2 + th + t)}{(t+h+1)(t+1)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{h}{(t+h+1)(t+1)}\right) \end{aligned}$$

$$\begin{aligned} g'(t) &= \lim_{h\to 0} \frac{1}{(t+h+1)(t+1)} \\ &= \frac{1}{(t+1)(t+1)} \\ &= \frac{1}{(t+1)^2} \end{aligned}$$

The derivative is then,

$$g'(t) = \frac{1}{(t+1)^2}$$

Example 3:

Find the derivative of the following function using the definition of the derivative.

$$R(z) = \sqrt{5z-8}$$

Solution:

$$\begin{aligned} R'(z) &= \lim_{h\to 0} \frac{R(z+h)-R(z)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{5(z+h)-8}-\sqrt{5z-8}}{h} \end{aligned}$$

$$\begin{aligned} R'(z) &= \lim_{h\to 0} \frac{\left(\sqrt{5(z+h)-8}-\sqrt{5z-8}\right)}{h} \frac{\left(\sqrt{5(z+h)-8}+\sqrt{5z-8}\right)}{\left(\sqrt{5(z+h)-8}+\sqrt{5z-8}\right)} \\ &= \lim_{h\to 0} \frac{5z+5h-8-(5z-8)}{h\left(\sqrt{5(z+h)-8}+\sqrt{5z-8}\right)} \\ &= \lim_{h\to 0} \frac{5h}{h\left(\sqrt{5(z+h)-8}+\sqrt{5z-8}\right)} \end{aligned}$$

Again, after the simplification we have only h's left in the numerator. So, cancel the h and evaluate the limit.

$$\begin{aligned} R'(z) &= \lim_{h\to 0} \frac{5}{\sqrt{5(z+h)-8}+\sqrt{5z-8}} \\ &= \frac{5}{\sqrt{5z-8}+\sqrt{5z-8}} \\ &= \frac{5}{2\sqrt{5z-8}} \end{aligned}$$

And so we get a derivative of,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

Email: Hasan.muwafaq.gheni@uomus.edu.iq

2. Interpretation of the Derivative

Rate of Change

The first interpretation of a derivative is rate of change. This was not the first problem that we looked at in the Limits chapter, but it is the most important interpretation of the derivative. If $f(x)$ represents a quantity at any x then the derivative $f'(a)$ represents the instantaneous rate of change of $f(x)$ at $x = a$.

Example 4:

Suppose that the amount of water in a holding tank at t minutes is given by $V(t) = 2t^2 - 16t + 35$. Determine each of the following.

(a) Is the volume of water in the tank increasing or decreasing at $t = 1$ minute?

(b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?

(c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes?

(d) Is the volume of water in the tank ever not changing? If so, when?

Solution:

The derivative is.

$$V'(t) = 4t - 16 \qquad \text{OR} \qquad \frac{dV}{dt} = 4t - 16$$

(a) Is the volume of water in the tank increasing or decreasing at $t = 1$ minute?

In this case all that we need is the rate of change of the volume at $t = 1$ or,

$$V'(1) = -12 \qquad \text{OR} \qquad \left.\frac{dV}{dt}\right|_{t=1} = -12$$

So, at $t = 1$ the rate of change is negative and so the volume must be decreasing at this time.

Email: Hasan.muwafaq.gheni@uomus.edu.iq

(b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?

Again, we will need the rate of change at $t = 5$.

$$V'(5) = 4 \qquad \text{OR} \qquad \left.\frac{dV}{dt}\right|_{t=5} = 4$$

In this case the rate of change is positive and so the volume must be increasing at $t = 5$.

(c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes?

To answer this question all that we look at is the size of the rate of change and we don't worry about the sign of the rate of change. All that we need to know here is that the larger the number the faster the rate of change. So, in this case the volume is changing faster at $t = 1$ than at $t = 5$.

(d) Is the volume of water in the tank ever not changing? If so, when?

The volume will not be changing if it has a rate of change of zero. In order to have a rate of change of zero this means that the derivative must be zero. So, to answer this question we will then need to solve

$$V'(t) = 0 \qquad \text{OR} \qquad \frac{dV}{dt} = 0$$

This is easy enough to do.

$$4t - 16 = 0 \qquad \Rightarrow \qquad t = 4$$

3. Differentiation Formulas

Properties of differentiation formulas

1. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$ OR $\frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx}$

2. $(cf(x))' = cf'(x)$ OR $\frac{d}{dx}(cf(x)) = c\frac{df}{dx}$, c is any number

Formulas

1. If $f(x) = c$ then $f'(x) = 0$ OR $\frac{d}{dx}(c) = 0$

2. If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ OR $\frac{d}{dx}(x^n) = nx^{n-1}$, n is any number.

Example 5:

Differentiate each of the following functions.

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

(b) $g(t) = 2t^6 + 7t^{-6}$

(c) $y = 8z^3 - \frac{1}{3z^5} + z - 23$

(d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \frac{2}{\sqrt[5]{x^2}}$

(e) $h(x) = x^{\pi} - x^{\sqrt{2}}$

Solution:

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

$$f'(x) = 15(100)x^{99} - 3(12)x^{11} + 5(1)x^{0} - 0$$
$$= 1500x^{99} - 36x^{11} + 5$$

(b) $g(t) = 2t^{6} + 7t^{-6}$

$$g'(t) = 2(6)t^{5} + 7(-6)t^{-7}$$
$$= 12t^{5} - 42t^{-7}$$

(c) $y = 8z^{3} - \dfrac{1}{3z^{5}} + z - 23$

$$y' = 24z^{2} + \frac{5}{3}z^{-6} + 1$$

(d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^{7}} - \dfrac{2}{\sqrt[5]{x^{2}}}$

$$T(x) = x^{\frac{1}{2}} + 9\left(x^{7}\right)^{\frac{1}{3}} - \frac{2}{\left(x^{2}\right)^{\frac{1}{5}}}$$
$$= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - \frac{2}{x^{\frac{2}{5}}}$$
$$= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - 2x^{-\frac{2}{5}}$$

$$T'(x) = \frac{1}{2}x^{-\frac{1}{2}} + 9\left(\frac{7}{3}\right)x^{\frac{4}{3}} - 2\left(-\frac{2}{5}\right)x^{-\frac{7}{5}}$$
$$= \frac{1}{2}x^{-\frac{1}{2}} + 21x^{\frac{4}{3}} + \frac{4}{5}x^{-\frac{7}{5}}$$

(e) $h(x) = x^{\pi} - x^{\sqrt{2}}$

In all of the previous examples the exponents have been nice integers or fractions. That is usually what we'll see in this class. However, the exponent only needs to be a number so don't get excited about problems like this one. They work exactly the same.

$$h'(x) = \pi x^{\pi-1} - \sqrt{2}x^{\sqrt{2}-1}$$

Example 6:

Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

(b) $h(t) = \dfrac{2t^5 + t^2 - 5}{t^2}$

Solution:

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

$$y = x^{\frac{2}{3}}\left(2x - x^2\right) = 2x^{\frac{5}{3}} - x^{\frac{8}{3}}$$

Now we can differentiate the function.

$$y' = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

(b) $h(t) = \dfrac{2t^5 + t^2 - 5}{t^2}$

$$h(t) = \frac{2t^5}{t^2} + \frac{t^2}{t^2} - \frac{5}{t^2} = 2t^3 + 1 - 5t^{-2}$$

This is a function that we can differentiate.

$$h'(t) = 6t^2 + 10t^{-3}$$

Example 7:

Is $f(x) = 2x^3 + \dfrac{300}{x^3} + 4$ increasing, decreasing or not changing at $x = -2$?

Solution:

$$f(x) = 2x^3 + 300x^{-3} + 4 \qquad \Rightarrow \qquad f'(x) = 6x^2 - 900x^{-4} = 6x^2 - \frac{900}{x^4}$$

$$f'(-2) = 6(4) - \frac{900}{16} = -\frac{129}{4} = -32.25$$

So, at $x = -2$ the derivative is negative and so the function is decreasing at $x = -2$.

Example 8:

Find the equation of the tangent line to $f(x) = 4x - 8\sqrt{x}$ at $x = 16$.

Solution:

We know that the equation of a tangent line is given by,

$$y = f(a) + f'(a)(x - a)$$

So, we will need the derivative of the function (don't forget to get rid of the radical).

$$f(x) = 4x - 8x^{\frac{1}{2}} \qquad \Rightarrow \qquad f'(x) = 4 - 4x^{-\frac{1}{2}} = 4 - \frac{4}{x^{\frac{1}{2}}}$$

Again, notice that we eliminated the negative exponent in the derivative solely for the sake of the evaluation. All we need to do then is evaluate the function and the derivative at the point in question, $x = 16$.

$$f(16) = 64 - 8(4) = 32 \qquad f'(16) = 4 - \frac{4}{4} = 3$$

The tangent line is then,

$$y = 32 + 3(x - 16) = 3x - 16$$

Email: Hasan.muwafaq.gheni@uomus.edu.iq