10 Divisibility Tests

Elementary school children know how to tell if a number is even, or divisible by 5, by looking at the least significant digit.

Theorem 10.1. *If a number a has the decimal representation*

$$a = a_{n-1}10^{n-1} + a_{n-2}10^{n-2} + \cdots + a_1 10 + a_0$$

then:

1. $a \mod 2 = a_0 \mod 2$
2. $a \mod 5 = a_0 \mod 5$

Proof. Consider the polynomial function:

$$f(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Note that $10 \equiv 0 \pmod 2$. So

$$a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0 \equiv a_{n-1}0^{n-1} + \cdots + a_1 0 + a_0 \pmod 2.$$

That is,

$$a \equiv a_0 \pmod 2.$$

This proves item (1).

Similarly, since $10 \equiv 0 \pmod 5$, the proof of item (2) follows in the same manner. □

Example 10.1. Thus, the number 1457 is odd because 7 is odd:

$$1457 \mod 2 = 7 \mod 2 = 1.$$

And on division by 5, it leaves a remainder of:

$$1457 \mod 5 = 7 \mod 5 = 2.$$

Theorem 10.2. *Let*

$$a = a_{n-1}10^{n-1} + a_{n-2}10^{n-2} + \cdots + a_1 10 + a_0$$

be the decimal representation of a*. Then:*

1. $a \mod 3 = (a_{n-1} + \cdots + a_0) \mod 3.$
2. $a \mod 9 = (a_{n-1} + \cdots + a_0) \mod 9.$
3. $a \mod 11 = (a_0 - a_1 + a_2 - a_3 + \dots) \mod 11.$

Proof. Note that $10 \equiv 1 \pmod 3$.

$$a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0 \equiv a_{n-1}1^{n-1} + \cdots + a_1 1 + a_0 \pmod 3.$$

Thus,

$$a \equiv a_{n-1} + \cdots + a_1 + a_0 \pmod 3.$$

This proves item (1). Similarly, since $10 \equiv 1 \pmod 9$, the proof of item (2) follows the same steps.

For item (3), note that $10 \equiv -1 \pmod{11}$, so:

$$a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0 \equiv a_{n-1}(-1)^{n-1} + \cdots + a_1(-1) + a_0 \pmod{11}.$$

That is,

$$a \equiv a_0 - a_1 + a_2 - a_3 + \dots \pmod{11}.$$

□

Example 10.2. Consider the number 1457. We calculate its remainder modulo 3, 9, and 11.

Modulo 3:

$$1457 \mod 3 = (1+4+5+7) \mod 3 = 17 \mod 3 = 8 \mod 3 = 2.$$

Modulo 9:

$$1457 \mod 9 = (1+4+5+7) \mod 9 = 17 \mod 9 = 8 \mod 9 = 8.$$

Modulo 11:

$$1457 \mod 11 = (7-5+4-1) \mod 11 = 5 \mod 11 = 5.$$

Thus, the least nonnegative residues are:

$$1457 \equiv 2 \pmod{3}, \quad 1457 \equiv 8 \pmod{9}, \quad 1457 \equiv 5 \pmod{11}.$$

Remark 10.1.

$$m \mid a \iff a \mod m = 0$$

Corollary 10.1. Let $a = a_{n-1}10^{n-1} + a_{n-2}10^{n-2} + \cdots + a_1 10 + a_0$. Then:

1. $2 \mid a \iff a_0 = 0, 2, 4, 6$, or 8.
2. $5 \mid a \iff a_0 = 0$ or 5.
3. $3 \mid a \iff 3 \mid (a_0 + a_1 + \cdots + a_{n-1})$.
4. $9 \mid a \iff 9 \mid (a_0 + a_1 + \cdots + a_{n-1})$.
5. $11 \mid a \iff 11 \mid (a_0 - a_1 + a_2 - a_3 + \dots)$.

Theorem 10.3. *Let* $a = a_r 10^r + \cdots + a_2 10^2 + a_1 10 + a_0$ *be the decimal representation, so that we write* a *as the sequence* $a_r a_{r-1} \ldots a_1 a_0$*. Then:*

1. $7 \mid a \iff 7 \mid (a_r \ldots a_1 - 2a_0)$.
2. $13 \mid a \iff 13 \mid (a_r \ldots a_1 - 9a_0)$,

where $a_r \ldots a_1$ *is the sequence representing* $\frac{a-a_0}{10}$.

Example 10.3. We can test whether 7 divides 2481:

$$7 \mid 2481 \iff 7 \mid (248 - 2) \iff 7 \mid 246 \iff 7 \mid (24 - 12) \iff 7 \mid 12.$$

Since $7 \nmid 12$, we conclude that $7 \nmid 2481$.

Example 10.4. The number 12987 is divisible by 13 because:

$$13 \mid 12987 \iff 13 \mid (1298 - 63) \iff 13 \mid 1235 \iff 13 \mid (123 - 45) \iff 13 \mid 78.$$

And since $13 \times 6 = 78$, we conclude that 12987 is divisible by 13.

10.1 Exercises of Divisibility Tests

Exercises

1. Let $a = 18726132117057$. Find $a \mod m$ for $m = 2, 3, 5, 9$, and 11.
2. Determine which of the following are divisible by 7:

   (a) 6994 (b) 6993
3. Let $a = a_n a_{n-1} \cdots a_1 a_0$ be the decimal representation of a. Prove the following:

   (a) $a \mod 10 = a_0$.

   (b) $a \mod 100 = a_1 a_0$.

   (c) $a \mod 1000 = a_2 a_1 a_0$.