

Al-Mustaqbal university
Engineering technical college
Department of Building
&Construction Engineering



Mathematics

First class

Lecture No. 2

Limits

Assist. Lecture

Alaa Hussein AbdUlameer

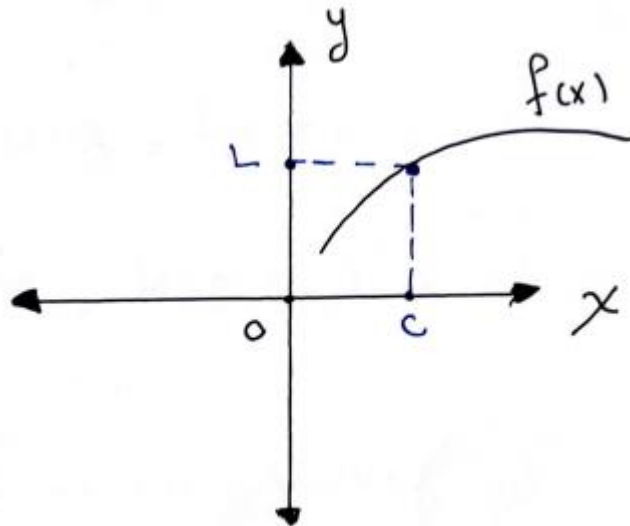
Assist. Lecture

Haneen Fadhel Kadhim

Limits:

When the values of a function $f(x)$ approach the value L as x Approaches C , we say that $f(x)$ has limit L as x Approaches C .

$$\text{OR: } \lim_{x \rightarrow c} f(x) = L$$



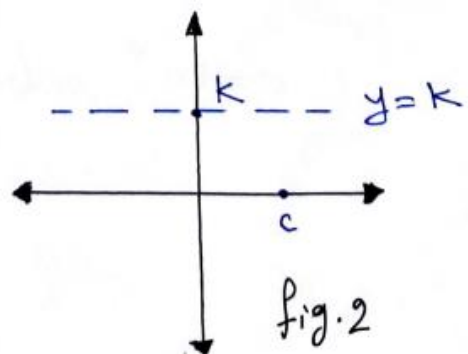
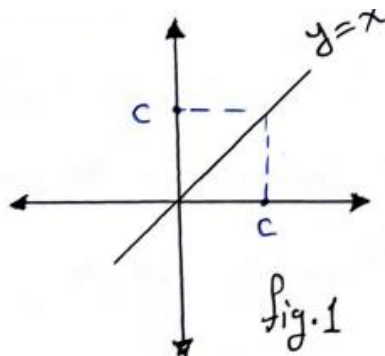
*Notes:

1. For identify function ($f(x)=x$)

$$\lim_{x \rightarrow c} x = c$$

2. For constant function ($f(x)=k$)

$$3. \lim_{x \rightarrow c} k = k$$



Properties of limits:

For

$$\lim_{x \rightarrow c} F_1(x) = L_1 \quad \& \quad \lim_{x \rightarrow c} F_2(x) = L_2$$

1. $\lim_{x \rightarrow c} [F_1(x) \pm F_2(x)] = L_1 \pm L_2$
2. $\lim_{x \rightarrow c} [F_1(x) \times F_2(x)] = L_1 \times L_2$
3. $\lim_{x \rightarrow c} [F_1(x) \div F_2(x)] = L_1 \div L_2 \quad L_2 \neq 0$
4. $\lim_{x \rightarrow c} [F_1(x) \times k] = k \times L_1 \quad k: constant$

- If $F_1(x) = a_n x^n + a_{n-1} x^{n-1} + \dots \dots \dots a_0$ is any polynomial function:
- $\lim_{x \rightarrow c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \dots \dots \dots a_0$

Ex (1): find the limit of the function $f(x)=x+1$ as x approaches 3?

Sol/

$$f(x)=x+1$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow 3} x + 1$$

$$\lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 1$$

$$3 + 1 = 4$$

Ex (2):

$$\lim_{x \rightarrow 3} \frac{x^3 - 27}{x - 3}$$

$$\frac{(0)^3 - 27}{0 - 3} = \frac{0}{0}$$

$$\lim_{x \rightarrow 3} \frac{(x-3)(x^2+3x+9)}{(x-3)}$$

$$\lim_{x \rightarrow 3} (x^2+3x+9)$$

$$(3^2+3(3)+9) = 27$$

Ex (3): Evaluate

$$\lim_{x \rightarrow 1} \frac{x^2+x-2}{x^2-x}$$

$$\frac{1^2+1-2}{1^2-1} = \frac{0}{0}$$

$$\lim_{x \rightarrow 1} \frac{(x+2)(x-1)}{x(x-1)}$$

$$\lim_{x \rightarrow 1} \frac{(x+2)}{x}$$

$$\frac{1+2}{1} = 3$$

Ex (4) find the limits:

$$\lim_{x \rightarrow 5} \frac{4}{x-7}$$

$$\frac{4}{5-7} = \frac{4}{-2} = -2$$

Ex (5) find the limits:

$$\lim_{x \rightarrow 2} \frac{x^2-7x-10}{x-2}$$

$$\lim_{x \rightarrow 2} \frac{(x-2)(x-5)}{(x-2)}$$

$$\lim_{x \rightarrow 2} (x-5) = 2-5 = -3$$

EX (6):

$$f(x) = \begin{cases} 3 - x & .x < 2 \\ \frac{x}{2} + 1 & .x > 2 \end{cases}$$

$$a - \text{find } \lim_{x \rightarrow 2} f(x) \text{ and } \lim_{x \rightarrow -2} f(x)$$

$$b - \text{dose } \lim_{x \rightarrow -2} f(x) \text{ exist? why}$$

Sol/

$$a - \lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x}{2} + 1 = \frac{2}{2} + 1 = 2$$

$$b - \lim_{x \rightarrow -2} f(x) = \lim_{x \rightarrow -2} (3 - x) = 3 - 2 = 1$$

$$\lim_{x \rightarrow 2} f(x) \neq \lim_{x \rightarrow -2} f(x)$$

Limit does not exist

Ex (7):

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{(x - 1)^2}$$

$$\lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x - 1)}$$

$$\lim_{x \rightarrow 1} \frac{(x^2 + x + 1)}{(x - 1)}$$

H.W/ Let:

$$f(x) = \begin{cases} 3 - x & .x < 2 \\ \frac{x}{2} & .x > 2 \end{cases}$$

$$a - \text{Find } \lim_{x \rightarrow 2} f(x) \text{ and } \lim_{x \rightarrow -2} f(x)$$

$$b - \text{dose } \lim_{x \rightarrow -2} f(x) \text{ exist? why}$$

Ex : find $\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x^3 + 8}$

$$\lim_{x \rightarrow -2} \frac{\frac{(x+2)}{2x}}{(x+2)(x^2 - 2x + 4)}$$

$$\begin{aligned} & \lim_{x \rightarrow -2} \frac{1}{2x(x^2 - 2x + 4)} \\ &= \frac{1}{2(-2)((-2)^2 - 2(-2) + 4)} \\ &= \frac{-1}{48} \end{aligned}$$

Ex:

$$\lim_{x \rightarrow 4} \frac{3 - \sqrt{x+5}}{x-4}$$

$$\lim_{x \rightarrow 4} \frac{3 - \sqrt{x+5}}{x-4} * \frac{3 + \sqrt{x+5}}{3 + \sqrt{x+5}}$$

$$\lim_{x \rightarrow 4} \frac{9 - (x+5)}{x-4(3 + \sqrt{x+5})}$$

$$\lim_{x \rightarrow 4} \frac{4-x}{x-4(3 + \sqrt{x+5})}$$

$$\lim_{x \rightarrow 4} \frac{-(x-4)}{(x-4)(3 + \sqrt{x+5})}$$

$$\lim_{x \rightarrow 4} \frac{-1}{(3 + \sqrt{x+5})} = \frac{-1}{(3 + \sqrt{4+5})} = \frac{-1}{6}$$

The sandwiches theorem

$f(x), h(x), g(x)$ are functions

$$\text{if } f(x) \leq h(x) \leq g(x)$$

The sandwiches theorem is:

$$\text{For all } x \neq c \text{ and } \lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$$

$$\text{Then } \lim_{x \rightarrow c} f(x) = L$$

$$\text{Ex (1): find } \lim_{x \rightarrow 0} x^4 \cos \frac{1}{x}$$

$$-1 \leq \cos \frac{1}{x} \leq 1$$

$$\left[-1 \leq \cos \frac{1}{x} \leq 1 \right] * x^4$$

$$-x^4 \leq x^4 \cos \frac{1}{x} \leq x^4$$

$$\lim_{x \rightarrow 0} -x^4 \leq \lim_{x \rightarrow 0} x^4 \cos \frac{1}{x} \leq \lim_{x \rightarrow 0} x^4$$

$$\lim_{x \rightarrow 0} -x^4 = (0)^4 = 0 \quad g(x)$$

$$\lim_{x \rightarrow 0} x^4 = (0)^4 = 0 \quad h(x)$$

$$0 \leq \cos \frac{1}{x} \leq 0$$

$$h(x) = g(x)$$

Cross ponding to sandwiches theory

$$\lim_{x \rightarrow 0} x^4 \cos \frac{1}{x} = 0 \quad f(x)$$

$$\text{Ex (2): find } \lim_{x \rightarrow 0} (x^2 \sin \frac{1}{\sqrt{x}})$$

Sol/

$$-1 \leq \sin \frac{1}{\sqrt{x}} \leq 1$$

$$\left[-1 \leq \sin \frac{1}{\sqrt{x}} \leq 1 \right] * x^2$$

$$-x^2 \leq x^2 \sin \frac{1}{\sqrt{x}} \leq x^2$$

$$\lim_{x \rightarrow 0} -x^2 \leq \lim_{x \rightarrow 0} x^2 \sin \frac{1}{\sqrt{x}} \leq \lim_{x \rightarrow 0} x^2$$

$$\lim_{x \rightarrow 0} -x^2 = -(0)^2 = 0 \quad g(x)$$

$$\lim_{x \rightarrow 0} x^2 = (0)^2 = 0 \quad h(x)$$

Sandwiches theory is:

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{\sqrt{x}} = 0 \quad f(x)$$

Ex (3): find $\lim_{x \rightarrow \infty} \frac{\cos x}{x^2 + 1}$

Sol/

$$-1 \leq \cos x \leq 1$$

$$[-1 \leq \cos x \leq 1] \div (x^2 + 1)$$

$$\frac{-1}{x^2 + 1} \leq \frac{\cos x}{x^2 + 1} \leq \frac{1}{x^2 + 1}$$

$$\lim_{x \rightarrow \infty} \frac{-1}{x^2 + 1} \leq \lim_{x \rightarrow \infty} \frac{\cos x}{x^2 + 1} \leq \lim_{x \rightarrow \infty} \frac{1}{x^2 + 1}$$

$$\lim_{x \rightarrow \infty} \frac{-1}{x^2 + 1} = \frac{-1}{\infty^2 + 1} = \frac{-1}{\infty^2} = 0 \dots g(x)$$

$$\lim_{x \rightarrow \infty} \frac{1}{x^2 + 1} = \frac{1}{\infty^2 + 1} = \frac{1}{\infty^2} = 0 \dots h(x)$$

By sandwiches theory

$$\lim_{x \rightarrow \infty} \frac{\cos x}{x^2 + 1} = 0 \dots f(x)$$

Ex (4):

Find $\lim_{x \rightarrow \infty} \frac{2 - \cos x}{x^2 + 3}$ by using sandwiches theorem?

Sol/

$$-1 \leq \cos x \leq 1$$

$$[-1 \leq \cos x \leq 1] * -1$$

$$1 \geq -\cos x \geq -1$$

$$[1 \geq -\cos x \geq -1] + 2$$

$$1 + 2 \geq 2 - \cos x \geq -1 + 2$$

$$[3 \geq 2 - \cos x \geq +1] \div (x^2 + 1)$$

$$\frac{3}{x^2 + 1} \geq \frac{2 - \cos x}{x^2 + 1} \geq \frac{+1}{x^2 + 1}$$

$$\lim_{x \rightarrow \infty} \frac{3}{x^2 + 1} \geq \lim_{x \rightarrow \infty} \frac{2 - \cos x}{x^2 + 1} \geq \lim_{x \rightarrow \infty} \frac{+1}{x^2 + 1}$$

$$\lim_{x \rightarrow \infty} \frac{3}{x^2 + 1} = \frac{3}{\infty^2 + 1} = \frac{3}{\infty^2} = 0 \dots \dots g(x)$$

$$\lim_{x \rightarrow \infty} \frac{+1}{x^2 + 1} = \frac{1}{\infty^2 + 1} = \frac{1}{\infty^2} = 0 \dots \dots h(x)$$

By using sandwiches:

$$\lim_{x \rightarrow \infty} \frac{2 - \cos x}{x^2 + 1} = 0 \dots \dots \dots f(x)$$

Ex (5): find $\lim_{x \rightarrow 4} g(x)$ for the function

$$\lim_{x \rightarrow 4} |g(x) - 5| \leq 3(x - 4)^2$$

Sol/

$$-3(x-4)^2 \leq g(x) - 5 \leq 3(x-4)^2$$

$$[-3(x-4)^2 \leq g(x) - 5 \leq 3(x-4)^2](+5)$$

$$5 - 3(x-4)^2 \leq g(x) \leq 5 + 3(x-4)^2$$

$$\lim_{x \rightarrow 4} 5 - 3(x-4)^2 = 5 - 3(4-4)^2 = 5 \dots \dots \dots g(x)$$

$$\lim_{x \rightarrow 4} 5 + 3(x-4)^2 = 5 + 3(4-4)^2 = 5 \dots \dots \dots h(x)$$

$$\therefore \lim_{x \rightarrow 4} g(x) = 5 \dots \dots \dots f(x)$$

H.W

$$1. \lim_{x \rightarrow \infty} \frac{5x^2 + \cos(7x-2)}{x^2+1}$$

$$2. \lim_{x \rightarrow \infty} \frac{\cos x + 2}{x+3}$$

$$3. \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x^2}$$

$$4. \lim_{x \rightarrow 0} 3 - x^2 \cos \frac{1}{x}$$

Exercises 1.2

Limit Calculations

Find the limits in Exercises 1–16.

1. $\lim_{x \rightarrow -7} (2x + 5)$
2. $\lim_{x \rightarrow 12} (10 - 3x)$
3. $\lim_{x \rightarrow 2} (-x^2 + 5x - 2)$
4. $\lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8)$
5. $\lim_{t \rightarrow 6} 8(t - 5)(t - 7)$
6. $\lim_{s \rightarrow 2/3} 3s(2s - 1)$
7. $\lim_{x \rightarrow 2} \frac{x + 3}{x + 6}$
8. $\lim_{x \rightarrow 5} \frac{4}{x - 7}$
9. $\lim_{y \rightarrow -5} \frac{y^2}{5 - y}$
10. $\lim_{y \rightarrow 2} \frac{y + 2}{y^2 + 5y + 6}$
11. $\lim_{x \rightarrow -1} 3(2x - 1)^2$
12. $\lim_{x \rightarrow -4} (x + 3)^{1984}$
13. $\lim_{y \rightarrow -3} (5 - y)^{4/3}$
14. $\lim_{z \rightarrow 0} (2z - 8)^{1/3}$
15. $\lim_{h \rightarrow 0} \frac{3}{\sqrt{3h + 1} + 1}$
16. $\lim_{h \rightarrow 0} \frac{5}{\sqrt{5h + 4} + 2}$

Find the limits in Exercises 17–30.

17. $\lim_{x \rightarrow 5} \frac{x - 5}{x^2 - 25}$
18. $\lim_{x \rightarrow -3} \frac{x + 3}{x^2 + 4x + 3}$
19. $\lim_{x \rightarrow -5} \frac{x^2 + 3x - 10}{x + 5}$
20. $\lim_{x \rightarrow 2} \frac{x^2 - 7x + 10}{x - 2}$
21. $\lim_{t \rightarrow 1} \frac{t^2 + t - 2}{t^2 - 1}$
22. $\lim_{t \rightarrow -1} \frac{t^2 + 3t + 2}{t^2 - t - 2}$
23. $\lim_{x \rightarrow -2} \frac{-2x - 4}{x^3 + 2x^2}$
24. $\lim_{y \rightarrow 0} \frac{5y^3 + 8y^2}{3y^4 - 16y^2}$
25. $\lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 - 1}$
26. $\lim_{v \rightarrow 2} \frac{v^3 - 8}{v^4 - 16}$
27. $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$
28. $\lim_{x \rightarrow 4} \frac{4x - x^2}{2 - \sqrt{x}}$
29. $\lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x + 3} - 2}$
30. $\lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$

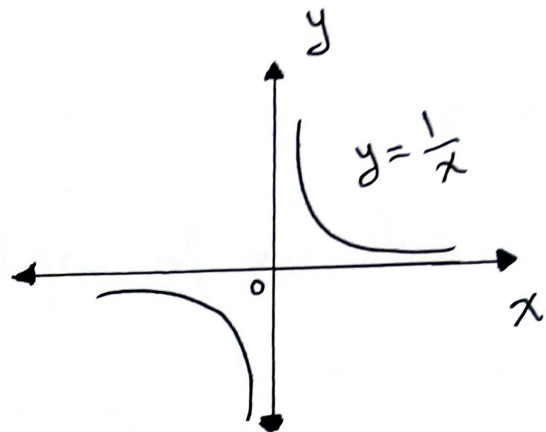
Lecture Two :

Limits Involving Infinity :-

For example, If we have

the function $y = \frac{1}{x}$

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$



Another example is a constant function $y = k$: $\lim_{x \rightarrow \infty} k = k$

$$\lim_{x \rightarrow -\infty} k = k$$

Note : the limit of $\frac{\sin \theta}{\theta}$ as θ approaches $\pm \infty$ is 0

Ex :- prove that $\lim_{\theta \rightarrow \infty} \frac{\sin \theta}{\theta} = 0$

Sol :-

$$-1 \leq \sin \theta \leq 1$$

$$[\div \theta] \quad \frac{-1}{\theta} \leq \frac{\sin \theta}{\theta} \leq \frac{1}{\theta}$$

$$\text{Now : } \lim_{x \rightarrow \infty} \frac{-1}{\theta} = 0 \quad \text{and} \quad \lim_{\theta \rightarrow \infty} \frac{1}{\theta} = 0$$

From Sandwich theorem $\Rightarrow \lim_{\theta \rightarrow \infty} \frac{\sin \theta}{\theta} = 0$

Limits of rational functions as x approaches $\pm\infty$.

Rules :- For the rational function $\frac{f(x)}{g(x)}$

1. If degree of $f(x)$ Less than degree of $g(x)$

$$\text{then } \lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = 0.$$

2. If degree of $f(x)$ equals degree of $g(x)$ then

$$\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} \text{ is finite.}$$

3. If degree of $f(x)$ greater than degree of $g(x)$

$$\text{then } \lim_{x \rightarrow \pm\infty} \text{ is infinite.}$$

- To solve the limit problem of rational functions as x approaches $\pm\infty$, divide both the numerator and the denominator by the highest power of x in denominator.

Ex :-

$$1. \lim_{x \rightarrow \infty} \frac{3x+1}{x^2-5} = 0 \quad \left[\begin{array}{l} \text{deg. of numerator less than} \\ \text{deg. of denominator} \end{array} \right]$$

$$\begin{aligned}
 2. \quad \lim_{x \rightarrow \infty} \frac{-x}{7x+3} &= \lim_{x \rightarrow \infty} \frac{-x/x}{(7x/x + 3/x)} \\
 &= \lim_{x \rightarrow \infty} \frac{-1}{7 + 3/x} = -1/7
 \end{aligned}$$

$$3. \quad \lim_{x \rightarrow \infty} \frac{-4x^3 + 7x}{2x^2 - 3x - 10}$$

$$\lim_{x \rightarrow \infty} \frac{(-4x^3/x^2) + (7x/x^2)}{(2x^2/x^2) - (3x/x^2) - (10/x^2)}$$

$$\lim_{x \rightarrow \infty} \frac{-4x + 7/x}{2 - 3/x - 10/x^2}$$

$$= \frac{-\infty + 0}{2 - 0 - 0} = \frac{-\infty}{2} = -\infty$$

$$4. \quad \lim_{x \rightarrow \infty} \frac{\cos 1/x}{1 + 1/x} = \frac{\cos 0}{1 + 0} = 1$$

$$5. \quad \lim_{x \rightarrow \infty} x \sin 1/x$$

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x}$$

$$= \lim_{\theta \rightarrow 0} \frac{1}{\theta} \sin \theta = 1$$

$$\text{let } \theta = \frac{1}{x}$$

$$\text{as } x \rightarrow \infty$$

$$\text{then } \theta \rightarrow 0$$