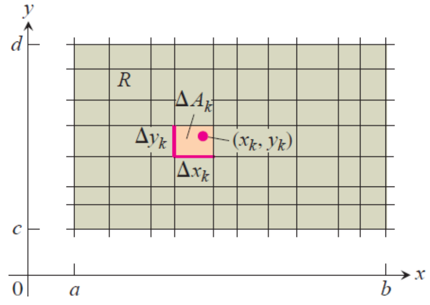
Chapter 7 Multiple Integral (Chapter 14)

Double Integral (14.1)



If $f(x,y)$ is defined on the rectangular region given by

$R: a \le x \le b, c \le y \le d$

Then we can write

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

The double integral represents the volume under the surface $z = f(x,y)$ within the region R

Ex: Evaluate the integral

$$\iint_R f(x,y)dA$$

Where $f(x,y) = 1 - 6x^2y$ and R: $0 \le x \le 2, -1 \le y \le 1$

Sol.

$$\iint_R f(x,y)dA = \int_{-1}^{1}\int_0^2 (1-6x^2y)dxdy = \int_{-1}^{-1} [x - 2x^3y]_0^2 dy$$

$$= \int_{-1}^{1} (2-16y)dy = 2y - 8y^2|_{-1}^{1} = 2 - 8 - (-2-8) = 4$$

This integral can be evaluated in the order of integration reversed

$$\iint_R f(x,y)dA = \int_0^2 \int_{-1}^{1} (1-6x^2y)dydx = \int_0^2 [y - 3x^2y^2]_{-1}^{1} dx$$

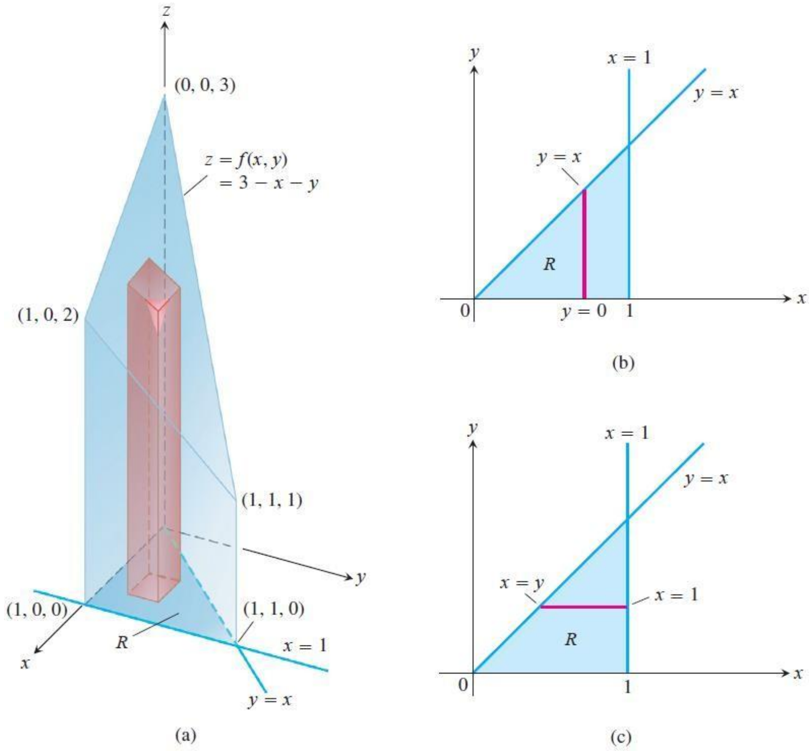
$$= \int_0^2 ((1-3x^2) - (-1-3x^2))dx = \int_0^2 (2)dx = 2x|_0^2 = 4$$

Double Integral Over Bounded Nonrectangular Regions

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Ex: Find the volume of the prism whose base is the triangle in the *xy*-plane bounded by the *x*-axis and the lines *y*=*x* and *x*=1 and whose top lies in the plane *z*=3–*x*–*y*.

Sol.



$$V = \int_0^1 \int_0^x (3 - x - y)\,dy\,dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2}\right) dx$$

$$= \int_0^1 \left(3x - \frac{3}{2}x^2\right) dx$$

$$= \left[\frac{3}{2}x - \frac{1}{2}x^3\right]_0^1 = \frac{3}{2} - \frac{1}{2} = 1$$

By reversing the order of integration

$$V = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_y^1 dy = \int_0^1 \left(\left(3 - \frac{1}{2} - y\right) - \left(3y - \frac{y^2}{2} - y^2\right)\right) dy$$

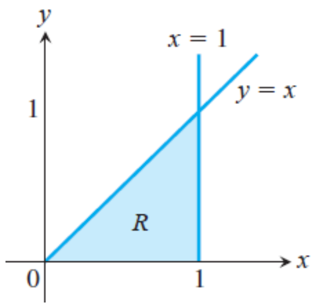
$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{1}{2}y^3\right]_0^1 = \frac{5}{2} - 2 + \frac{1}{2} = 1$$

Ex: Find

$$\iint_R \frac{\sin x}{x}\, dA$$

Where *R* is the triangular region in the *xy*-plane bounded by the lines *y*=*x*, *x*=1 and x-axis

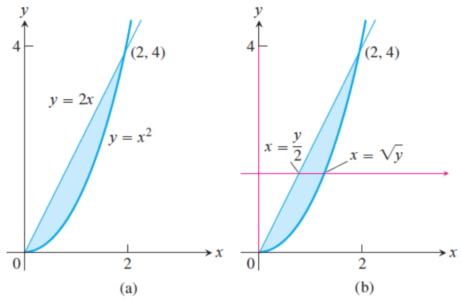
Sol.:



$$\int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx = \int_0^1 \left[y \frac{\sin x}{x}\right]_0^x = \int_0^1 \sin x = -\cos x|_0^1 = 1 - \cos 1$$

Ex: Find the value of the integration

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$$



With the order of the integration reversed

Sol.:

First it is better to find the points of intersection of the curves.

$x^2 = 2x \;\Rightarrow\; x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0 \;\Rightarrow y = 0$

$x = 2 \;\Rightarrow y = 4$

$y = x^2 \;\Rightarrow\; x = \sqrt{y}$

$y = 2x \;\Rightarrow\; x = y/2$

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy = \int_0^4 [2x^2+2x]_{y/2}^{\sqrt{y}} = \int_0^4 \left(2y + 2\sqrt{y} - \frac{y^2}{2} - y\right) dy$$

$$= \int_0^4 \left(y + 2\sqrt{y} - \frac{y^2}{2}\right) dy = \left[\frac{y^2}{2} + \frac{4}{3}y^{3/2} - \frac{y^3}{6}\right]_0^4 = 8 + \frac{32}{3} - \frac{32}{3} - 0 = 8$$