8. Integration methods/trigonometric substation: -

Formulas

$$\begin{pmatrix} \boldsymbol{a^2+u^2} \\ \boldsymbol{or} \\ \boldsymbol{\sqrt{a^2+u^2}} \end{pmatrix} \qquad \begin{pmatrix} \boldsymbol{a^2-u^2} \\ \boldsymbol{or} \\ \boldsymbol{\sqrt{a^2-u^2}} \end{pmatrix} \qquad \begin{pmatrix} \boldsymbol{u^2-a^2} \\ \boldsymbol{or} \\ \boldsymbol{\sqrt{u^2-a^2}} \end{pmatrix}$$

$\tan\theta$ $\sin\theta$ $\sec\theta$

Steps of solution

1. Assume $u = a\sin\theta \quad or\ u = a\tan\theta \quad or\ u = a\sec\theta$

2. We derive the two parties

3. After that we replace the assumed and derived function in the original function

Example 1: Find $\int \frac{1}{3^2+x^2}\,dx$ **.**

Solution //

1. $\boldsymbol{u = a\,tan\,\theta \rightarrow x = 3\,tan\,\theta \rightarrow \theta = \tan^{-1}\frac{x}{3}}$

2. $\boldsymbol{dx = 3\sec^2\theta\,d\theta}$

3. $=\int \frac{1}{3^2+(3\tan\theta)^2} 3\sec^2\theta\, d\theta =$

$\int \frac{1}{9(1+\tan^2\theta)} 3\sec^2\theta\, d\theta$

$$=\int \frac{\sec^2\theta}{3\sec^2\theta} d\theta = \frac{1}{3}\theta + c = \frac{1}{3}\tan^{-1}\frac{x}{3} + c$$

Example 2: *Find* $\int \frac{x^2}{\sqrt{9-x^2}}\, dx$.

Solution //

$$1.\ u = a\sin\theta \rightarrow x = 3\sin\theta \rightarrow \theta = \sin^{-1}\frac{x}{3}$$

$$2.\ dx = 3\cos\theta\, d\theta$$

3. $=\int \frac{(3\sin\theta)^2\, 3\cos\theta}{\sqrt{9-(3\sin\theta)^2}} d\theta = \int \frac{(3\sin\theta)^2\, 3\cos\theta}{\sqrt{9(1-\sin^2\theta)}} d\theta =$

$\frac{27\sin^2\theta\ \cos\theta}{3\sqrt{1-\sin^2\theta}} = 9\int \sin^2\theta\ d\theta = 9\int \frac{1}{2}(1-\cos 2\theta) d\theta =$

$\frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + c$

$$=\frac{9}{2}\theta - \frac{9}{4}2\ \sin\theta\cos\theta + c$$

$$=\frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{9}{2} * \frac{x}{3} * \frac{\sqrt{9-x^2}}{3} + c$$

Example 3: Find $\int \frac{1}{\sqrt{e^{2x}-1}} dx$

Solution// $\int \frac{1}{\sqrt{(e^x)^2-1}} dx$

1. $u = a\,sec\,\theta \rightarrow e^x = sec\,\theta \rightarrow \theta = \sec^{-1} e^x$

2.$e^x dx = \sec\theta\ \tan\theta\, d\theta$

3. $= \int \frac{sec\theta\ \ \tan\theta}{e^x\sqrt{sec^2\theta-1}} d\theta =$
$\int \frac{sec\theta\,.tan\,\theta}{sec\theta\sqrt{sec^2\theta-1}} d\theta = \int \frac{sec\theta\,.tan\,\theta}{sec\theta\,.tan\,\theta} d\theta = \theta + c =$
$sec^{-1} e^x + c$

H.W. //**1**. $\int e^x\sqrt{1-e^{2x}}dx$

2. $\int \frac{1}{4+x^2} dx$

3.$\int \frac{1}{x\sqrt{x^2-1}} dx$