# Lecture 7
# Hypothesis testing for a single population mean

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# 1. Setup

Test hypotheses about a single population mean $\mu$:

- Two-sided: $H_0 : \mu = \mu_0$ vs $H_a : \mu \neq \mu_0$.
- Right one-sided: $H_0 : \mu = \mu_0$ vs $H_a : \mu > \mu_0$.
- Left one-sided: $H_0 : \mu = \mu_0$ vs $H_a : \mu < \mu_0$.

Choose significance level $\alpha$ (common: 0.10, 0.05, 0.01).

## 2. Test statistics

Case Z (σ known):

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}, \qquad Z \sim N(0,1) \text{ under } H_0.$$

Case T (σ unknown):

$$T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}, \qquad T \sim t_{df=n-1} \text{ under } H_0.$$

# 3. Critical-value decision rules

Let the observed statistic be $z$ (or $t$).

**Two-sided** ($H_a : \mu \neq \mu_0$):

Reject $H_0$ if $|\text{statistic}| > \text{critical}_{1-\alpha/2}$.

- For z: critical $= z_{1-\alpha/2}$ (e.g. $\alpha = 0.05 \Rightarrow z_{0.975} = 1.96$).
- For t: critical $= t_{1-\alpha/2,\, df}$ from t-tables.

**Right one-sided** ($H_a : \mu > \mu_0$):

Reject $H_0$ if $\text{statistic} > \text{critical}_{1-\alpha}$.

- z: critical $= z_{1-\alpha}$ (e.g. $\alpha = 0.05 \Rightarrow z_{0.95} = 1.645$).
- t: critical $= t_{1-\alpha,\, df}$.

**Left one-sided** ($H_a : \mu < \mu_0$):

Reject $H_0$ if $\text{statistic} < -\text{critical}_{1-\alpha}$ (equivalently compare to negative critical).

# 4. Assumptions

1. Random sample (or approximate randomness).
2. Independence of observations.
3. For z: population normal or large $n$. For t: population approximately normal (t is robust for moderate departures when $n$ not tiny).

## Example 1 — z test, two-sided

Data: $n = 50,\ \bar{x} = 102,\ \sigma = 8.$

Test $H_0 : \mu = 100$ vs $H_a : \mu \neq 100,\ \alpha = 0.05.$

SE: $\sigma/\sqrt{n} = 8/\sqrt{50} \approx 8/7.0711 \approx 1.1314.$

Statistic: $Z = (102 - 100)/1.1314 = 2/1.1314 \approx 1.7678.$

Critical (two-sided, $\alpha = 0.05$): $z_{0.975} = 1.96.$

Compare: $|Z| = 1.7678 < 1.96$ ⇒ **Do not reject** $H_0$.

Conclusion: At the 5% level there is insufficient evidence that $\mu$ differs from 100.

## Example 2 — z test, right one-sided

Data: $n = 36,\ \bar{x} = 52,\ \sigma = 10.$

Test $H_0 : \mu = 50$ vs $H_a : \mu > 50,\ \alpha = 0.05.$

SE: $10/\sqrt{36} = 10/6 = 1.666666\ldots$.

Statistic: $Z = (52 - 50)/1.6666\ldots = 2/1.6666\ldots = 1.2.$

Critical (right one-sided, $\alpha = 0.05$): $z_{0.95} = 1.645.$

Compare: $Z = 1.2 < 1.645$ ⇒ **Do not reject** $H_0$.

Conclusion: Not enough evidence that $\mu > 50$ at 5%.

## Example 3 — z test, left one-sided (α=0.01)

Data: $n = 100,\ \bar{x} = 48,\ \sigma = 5.$

Test $H_0 : \mu = 50$ vs $H_a : \mu < 50,\ \alpha = 0.01.$

SE: $5/\sqrt{100} = 5/10 = 0.5.$

Statistic: $Z = (48 - 50)/0.5 = -2/0.5 = -4.0.$

Critical (left one-sided, $\alpha = 0.01$): lower critical $-z_{0.99}$. $z_{0.99} \approx 2.33$ so reject if $Z < -2.33.$

Compare: $Z = -4.0 < -2.33$ ⇒ **Reject** $H_0$.

Conclusion: Strong evidence that $\mu < 50$ at 1% level.

## Example 4 — t test, two-sided

Data: $n = 25,\ \bar{x} = 102,\ s = 6.$

Test $H_0 : \mu = 100$ vs $H_a : \mu \neq 100,\ \alpha = 0.05.$

SE: $s/\sqrt{n} = 6/5 = 1.2.$

Statistic: $T = (102 - 100)/1.2 = 2/1.2 = 1.666666\ldots$ (≈1.6667).

df = 24.

Critical (two-sided, $\alpha = 0.05$): $t_{0.975,24} \approx 2.064.$

Compare: $|T| = 1.6667 < 2.064$ ⇒ **Do not reject $H_0$.**

Conclusion: Insufficient evidence at 5% that $\mu \neq 100.$

## Example 5 — t test, right one-sided (small n)

Data: $n = 16,\ \bar{x} = 7.1,\ s = 2.5.$

Test $H_0 : \mu = 6.5$ vs $H_a : \mu > 6.5,\ \alpha = 0.05.$

SE: $2.5/\sqrt{16} = 2.5/4 = 0.625.$

Statistic: $T = (7.1 - 6.5)/0.625 = 0.6/0.625 = 0.96.$

df = 15.

Critical (right one-sided, $\alpha = 0.05$): $t_{0.95,15} \approx 1.753.$

Compare: $T = 0.96 < 1.753$ ⇒ **Do not reject $H_0$.**

Conclusion: Not enough evidence that $\mu > 6.5.$

## Example 6 — t test, two-sided (α=0.01)

Data: $n = 9,\ \bar{x} = 18,\ s = 4.$

Test $H_0 : \mu = 15$ vs $H_a : \mu \neq 15,\ \alpha = 0.01.$

SE: $4/\sqrt{9} = 4/3 = 1.333333\ldots.$

Statistic: $T = (18 - 15)/1.3333\ldots = 3/1.3333\ldots = 2.25.$

df = 8.

Critical (two-sided, $\alpha = 0.01$): $t_{0.995,8} \approx 3.355.$

Compare: $|T| = 2.25 < 3.355$ ⇒ **Do not reject** $H_0$.

Conclusion: At 1% level, evidence is not sufficient to say $\mu \neq 15.$

## Example 7 — z test, two-sided (large n)

Data: $n = 200,\ \bar{x} = 78,\ \sigma = 12.$

Test $H_0 : \mu = 75$ vs $H_a : \mu \neq 75,\ \alpha = 0.05.$

SE: $12/\sqrt{200} \approx 12/14.1421 \approx 0.8485.$

Statistic: $Z = (78 - 75)/0.8485 = 3/0.8485 \approx 3.536.$

Critical (two-sided, $\alpha = 0.05$): $z_{0.975} = 1.96.$

Compare: $|Z| = 3.536 > 1.96$ ⇒ **Reject** $H_0$.

Conclusion: Strong evidence that $\mu \neq 75$ (sample mean 78 significantly different).

## Example 8 — t test, right one-sided (moderate n, α=0.10)

Data: $n = 40,\ \bar{x} = 20.5,\ s = 3.2.$

Test $H_0 : \mu = 20$ vs $H_a : \mu > 20,\ \alpha = 0.10.$

SE: $3.2/\sqrt{40} \approx 3.2/6.3249 \approx 0.5057.$

Statistic: $T = (20.5 - 20)/0.5057 = 0.5/0.5057 \approx 0.989.$

df = 39.

Critical (right one-sided, $\alpha = 0.10$): $t_{0.90,39} \approx 1.303.$

Compare: $T = 0.989 < 1.303$ ⇒ **Do not reject $H_0$.**

Conclusion: Not enough evidence at 10% that $\mu > 20.$

## Example 9 — t test, two-sided (borderline large effect)

Data: $n = 30,\ \bar{x} = 105,\ s = 5.$

Test $H_0 : \mu = 100$ vs $H_a : \mu \neq 100,\ \alpha = 0.05.$

SE: $5/\sqrt{30} \approx 5/5.4772 \approx 0.9129.$

Statistic: $T = (105 - 100)/0.9129 \approx 5/0.9129 \approx 5.475.$

df = 29.

Critical (two-sided, $\alpha = 0.05$): $t_{0.975,29} \approx 2.045.$

Compare: $|T| = 5.475 > 2.045$ ⇒ **Reject** $H_0$.

Conclusion: Strong evidence that $\mu \neq 100$ (sample mean 105 is significantly higher).

## Example 10 — z test, right one-sided (small effect)

Data: $n = 49,\ \bar{x} = 101,\ \sigma = 4.$

Test $H_0 : \mu = 100$ vs $H_a : \mu > 100,\ \alpha = 0.05.$

SE: $4/\sqrt{49} = 4/7 = 0.571428\ldots$

Statistic: $Z = (101 - 100)/0.571428\ldots = 1/0.571428\ldots = 1.75.$

Critical (right one-sided, $\alpha = 0.05$): $z_{0.95} = 1.645.$

Compare: $Z = 1.75 > 1.645$ ⇒ **Reject** $H_0$.

Conclusion: At 5% level we conclude $\mu > 100$ (small but statistically significant effect).