Homogenous first-Order Differential Equation

An equation of the form :-

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \quad \text{———} (1)$$

is called a homogenous equation. In this equation, the variables cannot be separated but can be transformed by a change of variable into an equation where the variables can be separated.

To transform a homogenous equation such as equation (1) into an equation whose variables may separated we introduce the new independent variable :-

$$y = u \cdot x \quad \Rightarrow \quad dy = u \cdot dx + x\, du$$

عند اجراء هذا التعويض فان المعادله التفاضلية دائماً تتحول الى داله قابلة للفصل وبذلك يسهل حلها . وبعد الحصول على الحل نعوض بدل كل u ب :

$$u = \frac{y}{x}.$$

تكون المعادله التفاضليه متجانسه اذا كانت الداله $M(x,y)$ والداله $N(x,y)$ متجانسه ايضاً وبنفس الدرجة ويتم التحقق من ذلك من خلال تعويض عن كل x ب λx وعن كل y ب λy فتحقق ما يلي :-

$$\left.\begin{aligned} M(\lambda x, \lambda y) &= \lambda \cdot M(x,y) \\ N(\lambda x, \lambda y) &= \lambda \cdot N(x,y) \end{aligned}\right\} \text{This part is homogenous.}$$

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Ex(1): Find the general solution for $(x^2+y^2)\,dx + xy\,dy = 0$

$$(x^2+y^2)\,dx = -xy\,dy$$

$$M(x,y) = x^2+y^2$$

$$M(\lambda x,\lambda y) = \lambda^2 x^2 + \lambda^2 y^2 = \lambda^2(x^2+y^2)$$

$$= \lambda^2\, M(x,y). \quad \therefore \text{homogenous.}$$

$$N(x,y) = -xy$$

$$N(\lambda x,\lambda y) = -(\lambda x \cdot \lambda y) = -\lambda^2 (xy)$$

$$= \cancel{-}\lambda^2\, N(x,y). \quad \therefore \text{homogenous.}$$

To solve this equation, assume

$$y = u \cdot x \Rightarrow dy = u\,dx + x\,du$$

$$(x^2 + u^2x^2)\,dx = -x\,(u \cdot x)\,(u\,dx + x\,du)$$

$$x^2 dx + u^2x^2 dx = -(x^2u^2\,dx + x^3u\,du).$$

$$(x^2+2x^2u^2)\,dx = -x^3u\,du$$

$$(1+2u^2)\cancel{x^2}\,dx = -x^{\cancel{3}}u\,du$$

$$\frac{dx}{x} = \frac{-u}{1+2u^2}\,du \quad \Big\}\ \text{by integral}$$

$$\ln x = -\tfrac{1}{4}\ln(1+2u^2) + c \quad \Big]*4$$

$$4\ln x + \ln(1+2u^2) = C_2 \qquad \boxed{C_2 = 4C}$$

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Ex(2): Find the general solution for $y^2dx + x^2dy = 2xy\,dy$?

$$y^2\,dx = (2xy - x^2)\,dy \quad —(*)$$

$$M(x,y) = y^2$$
$$M(\lambda x, \lambda y) = \lambda^2 y^2$$
$$= \lambda^2 M(x,y) \quad \therefore \text{homogenous.}$$

$$N(x,y) = 2xy - x^2$$
$$N(\lambda x, \lambda y) = 2\lambda x \cdot \lambda y - \lambda^2 x^2 = \lambda^2 \cdot (2xy - x^2)$$
$$= \lambda^2 \cdot N(x,y) \quad \therefore \text{homogenous}$$

from eq. (*)

$\therefore$ The total equation is homogenous.

$$\frac{dy}{dx} = \frac{y^2}{2xy - x^2} \Rightarrow$$ To solve this equation, assume:-

$$y = u \cdot x \quad \text{and} \quad dy = u \cdot dx + x \cdot du$$

$$u^2x^2\,dx + x^2(u \cdot dx + x \cdot du) = 2x\,(u \cdot x)\,du.$$
$$u^2x^2\,dx + x^2u\,dx = 2ux^3\,du - x^3\,du$$
$$(u^2 + u)x^2\,dx = x^3(2u - 1)\,du$$
$$\frac{\cancel{x^2}}{x^{\cancel{3}}}dx = \left(\frac{2u-1}{u+u^2}\right)du \quad = \frac{1}{x}dx = \left(\frac{2u-1}{u+u^2}\right)du$$
$$\ln x = \ln(u+u^2) + c_1 \Longrightarrow \ln x - \ln(u+u^2) = c_1$$

$$\ln\frac{x}{u+u^2} = c_1 \quad \Big\}$$ نأخذ e للطرفين

$$\frac{x}{u+u^2} = e^c \Longrightarrow \frac{x}{u+u^2} = c \qquad [c = e^x].$$

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but $u = \frac{y}{x}$ lead to

$$\frac{x}{\frac{y}{x} + \frac{y^2}{x^2}} = c \Rightarrow \boxed{x^3 - y(x-y)c = 0} \quad G.E$$

Ex(3) :- Find the general solution for

$$x^2 dy + (y^2 - xy)dx = 0$$
$$x^2 dy = -(y^2 - xy)dx$$

$$M(x,y) = -(y^2 - xy)$$
$$M(\lambda x, \lambda y) = -(\lambda^2 y^2 - \lambda x \lambda y)$$
$$= -\lambda^2(y^2 - xy)$$
$$= \lambda^2 M(x,y) \quad \therefore \text{homogenous.}$$

$$N(x,y) = x^2$$
$$N(\lambda x, \lambda y) = \lambda^2 x^2$$
$$= \lambda^2 N(x,y) \quad \therefore \text{homogenous.}$$

$\therefore$ The total equation is homogenous.

$$\frac{dy}{dx} = -\frac{(y^2 - xy)}{x^2} \Rightarrow$$ To solve this equation, assumes

$$y = u \cdot x \quad \text{and} \quad dy = u \cdot dx + x \cdot du$$

$$x^2(u \cdot dx + x \cdot du) = -(u^2 x^2 - x^2 u)dx$$
$$\cancel{x^2 u dx} + x^3 du + u^2 x^2 dx - \cancel{x^2 u dx} = 0$$
$$x^3 du + u^2 x^2 dx = 0$$
$$u^2 \cancel{x^2} dx = -x^{\cancel{3}} du$$
$$\frac{dx}{x} = \frac{-1}{u^2} du \Rightarrow \int \frac{1}{x} dx = -\int \frac{1}{u^2} du$$

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$$\ln x + \frac{1}{u} = c \Rightarrow \boxed{\ln x + \frac{x}{y} = c}.$$

Ex(4):- Find the general solution for

$$(x+y)\,dy + (x-y)\,dx = 0$$

$$M(x,y) = (x-y)$$

$$M(\lambda x,\lambda y) = (\lambda x - \lambda y) \Rightarrow \lambda(x-y)$$

$$\therefore M(\lambda x,\lambda y) = \lambda\, M(x,y)$$

$\therefore$ homogenous.

$$N(x,y) = (x+y)$$

$$N(x\lambda,y\lambda) = (\lambda x + \lambda y) = \lambda(x+y) = \lambda\, N(x,y)$$

$\therefore$ homogenous.

$\therefore$ The total equation is homogenous.

$$\frac{dy}{dx} = -\frac{(x-y)}{(x+y)} \Rightarrow \text{To solve this equation, assume}$$

$$y = u\cdot x \Rightarrow dy = u.dx + x.du$$

$$(x+ux)(u.dx + x.du) + (x-ux)\,dx = 0$$

$$xu\,dx + x^2 du + u^2 x\,dx + ux^2 du + x\,dx - ux\,dx = 0$$

$$(\cancel{xu} + u^2 x \cancel{- ux} + x)\,dx + (x^2 + ux^2)\,du = 0$$

$$(u^2 x + x)\,dx + (1+u)x^2\,du = 0$$

$$(u^2+1)\cancel{x}\,dx + (1+u)\,x^{\cancel{2}}\,du = 0$$

$$\frac{1}{x}\,dx + \frac{1+u}{1+u^2}\,du = 0$$

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$$\frac{1}{x}\,dx + \frac{1}{1+u^2}\,du + \frac{u}{1+u^2}\,du = 0 \Big\}\text{ by integral}$$

$$\ln x + \tan^{-1} u + \frac{1}{2}\ln(1+u^2) + c = 0$$

$$\boxed{\ln x + \tan^{-1}\frac{y}{x} + \frac{1}{2}\ln\left(1+\frac{y}{x}\right) + c = 0}\,.$$

H.W: Find the general solution for

1- $(xy+y^2)\,dx = (x^2+xy+y^2)\,dy$.

Ans. $\ln x + \dfrac{1}{2\frac{y^2}{x^2}} - \dfrac{x}{y} + \ln\dfrac{y}{x} + c = 0$

2- $(3y^3 - x^3)\,dx = 3xy^2\,dy$. $\quad y(1) = 2$

Ans. $\ln x + \dfrac{1}{3}\dfrac{y^3}{x^3} = \dfrac{8}{3}$.

3- $\dfrac{dy}{dx} = \dfrac{y+2}{x+y+1}$ Ans.

$\ln(x+1) + \dfrac{x-1}{y+2} + \ln\dfrac{y+2}{x-1} = c$

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