Chapter One
Differential Equations

* Definition of Differential Equations :-

Is any equation which contains one derivative or more. The derivative may be either ordinary derivative or partial derivative, such as :-

$$\frac{dy}{dx} + \frac{d^2y}{dx^2} = \cos x \quad , \quad y'' + y' - \ln x = 0$$

* Classification of differential equations :-

— By Types :-

— Ordinary differential equation (ODE) :- in which all derivatives are with respect to a single independent variable, (Independent variable = 1), such as :-

$$\frac{dy}{dx} + \ln x = x \quad , \quad \frac{dy}{dx} + \frac{dz}{dx} = 0 .$$

— partial differential equation (PDE) :-

In which at least one derivative is with respect to two or more indepent variables, (Independent variable > 1,

E-mail : marwan.abbas.madhloom@uomus.edu.iq

such as:-

$z = x^2 + y^2$

$\frac{dz}{dx} = 2x \quad , \quad \frac{dz}{dy} = 2y$

$z' = 2x \quad , \quad z' = 2y$

$z = f(x, y)$

dependent variable. independent variable.

— By order:-

The order of the differential equation is the order of the highest derivative appears in that equation, for example:-

$(\frac{dy}{dx})^2 + \sin x = 0$ is ~~a~~ first-order ordinary differential equation (1^{st} order ODE)

$(\frac{\partial^3 u}{\partial x^3}) + \frac{\partial u}{\partial y} = 0$ is a third-order partial differential equation (3^{rd} order PDE).

— By Degree:

The highest power which is raised to the highest-order derivative existed in differential equation, for example:-

$\left(\frac{dy}{dx}\right)^2 + \sin x = 0$ is a 2^{nd} degree, 1^{st} order ODE.

$y'' + y' - y^2 = e^x$ is a 1^{st} degree, 2^{nd} order ODE.

$(y''')^2 + 2(y')^4 = 2x$ is a 2^{nd} degree, 3^{rd} order ODE.

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Note :- If highest-order derivative found inside the root or any form of fraction, the degree of this differential equation cannot be determine (undefined degree)

For example :-

$\sqrt{y''}$, $\sin y''$, yy''

$$\sqrt{\frac{d^2y}{dx^2}} + \frac{dy}{dx} = x . \quad \text{(undefined degree)}.$$

$$\frac{d^2y}{dx^2} + \cos\left(\frac{d^3y}{dx^3}\right) = 5x \quad \text{(undefined degree)}.$$

— By Linearity :-

A differential equation is said to be "linear DE" if and only if each term of the equation which contains a dependent variable and/or its derivative is of linear form. In another words a differential equation is said to be "linear DE" if :-

1- The dependent variable and all its derivatives appear in a linear form.

2- There is no production of a dependent variable with one of its derivatives, or one of its derivatives with another derivative,

For example :-

$y''' + 2y' + y = x^2$ is a linear, 1^{st} degree, 3^{rd} order ODE.

$\frac{dy}{dx} + y^2 = 1$ is a non linear, 1^{st} degree, 1^{st} order ODE.

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$\frac{d^2y}{dx^2} + \sin y = 0$ is a non-linear, 1^{st} degree, 2^{nd} order ODE.

$y^{iv} + (y')^2 = x$ is a non-linear, 1^{st} degree, 4^{th} order ODE.

$\frac{\partial^3 u}{\partial x^3} = u \cdot \frac{\partial u}{\partial y}$ is a non-linear, 1^{st} degree, 3^{rd} order PDE.

$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} \cdot \frac{dy}{dx} = 0$ is a non-linear, 1^{st} degree, 4^{th} order ODE.

First-Order Ordinary Differential Equations

A differential equation of the first order and first degree may be written in one of the following two forms:

* $\frac{dy}{dx} = f(x,y)$ (The slope form).

* $M(x,y)\,dx + N(x,y)\,dy = 0$ (The exact form).

For example, the following DE

$$\frac{dy}{dx} = \frac{2x-y}{y-3x}$$ may be rewritten as

$$(2x-y)\,dx - (y-3x)\,dy = 0,$$

where $f(x,y) = \frac{2x-y}{y-3x}$, $M(x,y) = 2x-y$, and $N(x,y) = -(y-3x)$.

→ **Separable variable differential equations:**

If we can separate the dependent variable and its differential

E-mail : marwan.abbas.madhloom@uomus.edu.iq

in a DE, then this DE is called "separable variables DE". For example, if a given first-order DE can be reduced to

$$\frac{dy}{dx} = f(x,y) \Rightarrow \frac{dy}{dx} = g(x)\cdot h(y)$$

$$\Rightarrow \frac{1}{h(y)}\,dy = g(x)\,dx.$$

or $\quad M(x,y)\,dx + N(x,y)\,dy = 0$

$$g_1(x)\,h_1(y)\,dx + g_2(x)\,h_2(y)\,dy = 0$$

$$\frac{g_1(x)}{g_2(x)}\cdot dx + \frac{h_2(y)}{h_1(y)}\cdot dy = 0,$$

then such equation is called "separable variable DE" which can be solved by integrating both sides.

Ex(1): Solve the following first-order differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

$\frac{dy}{dx} = -\frac{x}{y}$ (separable variable DE)

$$\Rightarrow y\cdot dy = -x\,dx \quad \therefore \int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + c_1, \text{ or } \Rightarrow y^2 = -x^2 + 2c_1$$

$$\therefore x^2 + y^2 = 2c_1 \Rightarrow \boxed{2c_1 = c} \Rightarrow x^2 + y^2 = c$$

(General solution G.S)

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As a check, we try to find dy/dx by differentiating the general solution,

$$x^2+y^2=c \Rightarrow 2x\,dx+2y\,dy=0 \Rightarrow \frac{dy}{dx}=\frac{-x}{y}$$

$\therefore$ ok

Ex(2): Solve $(1+x^3)\cdot dy - x^2 y\,dx = 0$

$(1+x^3)\cdot dy - x^2 y\,dx$ (separable variables DE)

$$\Rightarrow \frac{dy}{y}-\frac{x^2}{1+x^3}dx=0 \Rightarrow \int\frac{dy}{y}-\int\frac{x^2}{1+x^3}dx=0$$

$$\ln y-\frac{1}{3}\ln(1+x^3)=c_1 \Rightarrow 3\ln y-\ln(1+x^3)=3c$$

$$\ln y^3-\ln(1+x^3)=c_2 \Rightarrow \boxed{c_2=3c_1},$$

$$\ln\frac{y^3}{(1+x^3)}=c_2 \Rightarrow \frac{y^3}{1+x^3}=e^{c_2} \Rightarrow \frac{y^3}{1+x^3}=c_3$$

$$\boxed{c_3=e^{c_2}}$$

$\therefore y^3=c_3(1+x^3)$. (G.S)

Ex(3): solve $y'=xy-x \qquad y(0)=3.$

$$\frac{dy}{dx}=xy-x \Rightarrow \frac{dy}{dx}=x(y-1)$$

separable variable DE

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$\therefore \frac{dy}{dx} = xy - x \Rightarrow \frac{dy}{dx} = x(y-1)$

$\therefore \frac{dy}{y-1} = x\,dx \Rightarrow \int \frac{dy}{y-1} = \int x\,dx$

$\ln(y-1) = \frac{x^2}{2} + c_1$ or $y - 1 = e^{x^2/2 + c_1}$

$y - 1 = e^{x^2/2} \cdot e^{c_1} \Rightarrow y - 1 = C\, e^{x^2/2}$ $\boxed{C = e^{c_1}}$

$\therefore y = 1 + C e^{x^2/2}$ (G.S)

Apply the given condition, at $x = 0$, $y = 3$

$\Rightarrow 3 = 1 + C e^{0^2/2} \Rightarrow C = 2$

$\therefore y = 1 + 2 e^{x^2/2}$. (P.S).

Ex (4):- solve $\frac{dy}{dx} = y^2 - 4$.

We put the equation in the form $\frac{dy}{y^2-4} = dx$

$\Rightarrow \frac{1}{y^2-4} = \left(\frac{1/4}{y-2} - \frac{1/4}{y+2}\right)$

$\therefore \left(\frac{1/4}{y-2} - \frac{1/4}{y+2}\right) dy = dx$

Integrating and using laws of logarithem gives.

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$$\int \left(\frac{1/4}{y-2} - \frac{1/4}{y+2}\right) dy = \int dx$$

$$\tfrac{1}{4}\ln(y-2) - \tfrac{1}{4}\ln(y+2) = x + c_1 \quad]\times 4.$$

$$\ln\frac{y-2}{y+2} = 4x + c_2 \qquad \boxed{c_2 = 4c_1}.$$

$$\frac{y-2}{y+2} = e^{4x+c_2} \Longrightarrow \frac{y-2}{y+2} = e^{4x} + c \qquad \boxed{c = e^{c_2}}$$

$$y-2 = (y+2)(e^{4x}+c)$$

$$y-2 = ye^{4x} + yc + 2e^{4x} + 2c$$

$$y - ye^{4x} + yc = 2e^{4x} + 2c + 2$$

$$y(1-e^{4x}+c) = 2(e^{4x}+c+1)$$

$$y = 2\left(\frac{1+e^{4x}+c}{1-e^{4x}+c}\right) \quad (G.S)$$

Ex(5): Solve $\dfrac{dy}{dx} = -2 + e^{2x+y-1}$

let $z = 2x+y-1 \Rightarrow dz = 2\,dx + dy$

$$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 2$$

$$\therefore \frac{dz}{dx} \cancel{-2} = e^{z} \cancel{-2} \Rightarrow \frac{dz}{dx} = e^{z}$$

(separable variable DE)

$$\frac{dz}{e^{z}} = dx \Rightarrow \int \frac{1}{e^{z}} dz = dx$$

$$-e^{-z} = x + c_1 \quad \text{or} \quad x + e^{-z} = -c_1$$

$$x + e^{-2x-y+1} = c \Rightarrow \boxed{c = -c_1} \quad G.S$$

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Ex(6): Solve $x(2xy+1)\,dy + y(1+2xy-x^3y^3)\,dx = 0$

$$z = xy \Rightarrow dz = x\,dy + y\,dx \Rightarrow dy = \frac{dz - y\,dx}{x}$$

$$dy = \frac{dz - \frac{z}{x}dx}{x} \quad \therefore \quad x(2z+1)\cdot\frac{dz - \frac{z}{x}dx}{x} + \frac{z}{x}(1+2z-z^3)\,dx = 0$$

$$(2z+1)\,dz - \frac{z^4}{x}\,dx = 0 \quad (\text{separable}).$$

$$\left(\frac{2z+1}{z^4}\right)dz - \frac{dx}{x} = 0 \Rightarrow \int \frac{2z+1}{z^4}\,dz = \frac{dx}{x}$$

$$\int \left[\frac{2}{z^3} + \frac{1}{z^4}\right] dz = \int \frac{dx}{x}$$

$$-\frac{1}{z^2} - \frac{1}{3z^3} - \ln x = C_1 \Rightarrow \frac{1}{(xy)^2} + \frac{1}{3(xy)^3} + \ln x = C$$

$$\boxed{C = C_1} \cdot (\text{G.S})$$

H.W: Find a general solution for:

(1) $y' = 3x^2(1+y)$ Ans. $y = ke^{x^3} - 1$.

(2) $(y^2-1)\,dx = xy\,dy$, if $x=2$, $y=0$

Ans. $x^2 + 4y^2 = 4$.

(3) $(y + x^2\cdot y)\,dy = (x\cdot y^2 + x)\,dx$

Ans. $y^2+1 = k(1+x^2)$

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