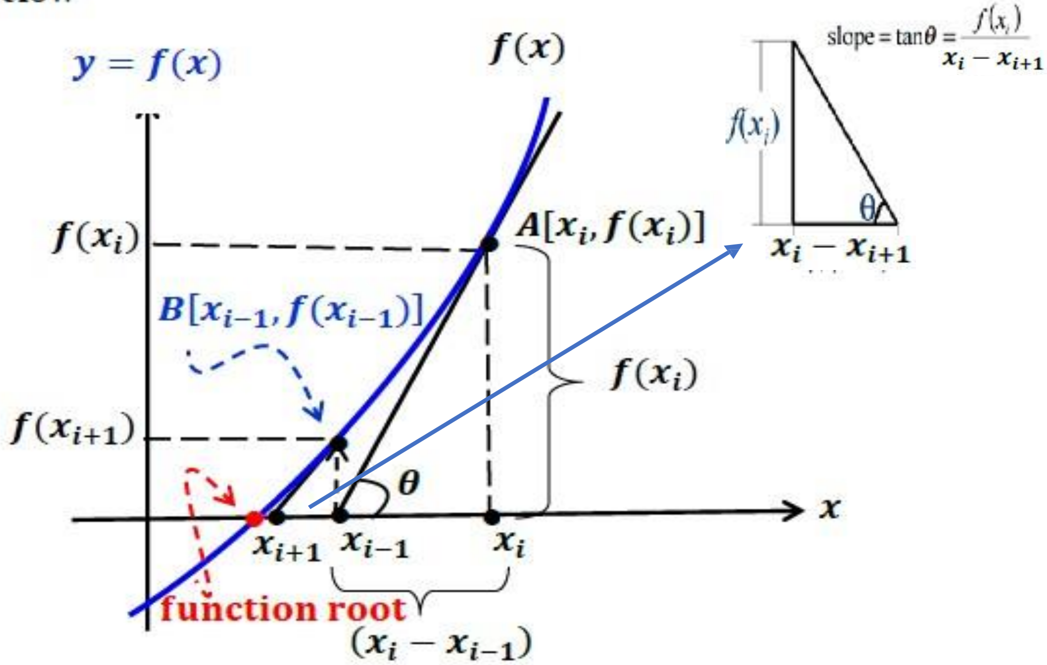
② Newton-Raphson Method

To **develop** or **derive** the **formula** of **Newton-Raphson method** for **finding** the **root** of the **nonlinear equation** $f(x) = 0$, **consider** the **function** $f(x)$ **shown below**



Starting with **some initial guess point** $A[x_i, f(x_i)]$ and **draw** the **tangent** at this **particular point. Extending** this **tangent** will **cross** the x-**axis** at **point** x_{i-1}. Then **go back** to the **function** to have **new guess point** $B[x_{i+1}, f(x_{i+1})]$ and **again draw tangent** at **this point**.

Keeping on drawing tangents, it is seen that **this tangent** is **eventually** going to **pass very close** to the **zero** of the **function** $f(x)$, or to the **root** where $f(x) = 0$, and this is what the **basis** of the **Newton-Raphson method** is.

Continually draw the **tangents** and see **where it crosses** the x-**axis** and **use it** as **new estimate** of the **root**.

Suppose that the **angle** of the **tangent** to **the function** $f(x)$ is θ, then from the figure,

$$\tan\theta = \frac{f(x_i)}{x_i - x_{i+1}}$$

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Since the **slope** of the **tangent line** at a **point** on the **function** is **equal** to the **derivative** of the **function** at the **same point**, thus

$$\tan\theta = f'(x_i) \quad \Longrightarrow \quad f'(x_i) = \tan\theta = \frac{f(x_i)}{x_i - x_{i+1}}$$

From this equation can be written

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

This equation is called the **Newton-Raphson formula** for finding the root of an equation.

Newton-Raphson Method Algorithm

The **algorithm** for **finding** the **root** of the **equation** $f(x) = 0$ is:

- **i.** Calculate $f'(x)$ symbolically.
- **ii.** Choose an **initial guess** (x_0).
- **iii.** Applying **Newton-Raphson formula**

$$x_1 = x_0 - \frac{f(x_o)}{f'(x_0)}$$

- **iv.** Find the **absolute value of relative approximate error**,

$$|\varepsilon_a| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100$$

- **v.** Check if **relative approximate error,** $|\varepsilon_a|$ **less than or equal** to **pre-specified tolerance,** ε_s

$$|\varepsilon_a| \le \varepsilon_s$$

- If $|\varepsilon_a| \le \varepsilon_s$, then the **process has to be stopped**
- **If not**, then **repeat this process again**

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Example: Use the **Newton-Raphson method** to **find** the **root** of the **equation** $x^3 = 20$ with **initial guess** $x_0 = 3$ and **conduct three iterations.**

Solution:

Re-write the function in the form

$$f(x) = x^3 - 20 = 0$$

The general form of the **Newton-Raphson method** is:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f(x) = x^3 - 20 \quad \Longrightarrow \quad f'(x) = 3x^2$$

Iteration 1:

Start with $x_0 = 3$, will have

$$\Longrightarrow \quad f(x_0) = (x_0)^3 - 20 - 20 = 3^3 - 20 = 27 - 20 = 7$$

$$\Longrightarrow \quad f'(x_0) = f'(3) = 3(x_0)^2 = 3 \times 3^2 = 27$$

$$\Longrightarrow x_1 = 3 - \frac{7}{27} = 2.741$$

The **absolute relative approximate error** is:

$$|\varepsilon_a| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100 = \left|\frac{2.741 - 3}{2.741}\right| \times 100 = 9.45\,\% > 5\%$$

Iteration 2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{(x_1)^3 - 20}{3(x_1)^2} = 2.741 - \frac{(2.741)^3 - 20}{3 \cdot (2.741)^2}$$

$$\Longrightarrow x_2 = 2.715$$

$$|\varepsilon_a| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100 = \left|\frac{2.715 - 2.741}{2.715}\right| \times 100 = 0.96\,\% > 0.5\%$$

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Iteration 3:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{(x_2)^3 - 20}{3(x_2)^2} = 2.715 - \frac{(2.715)^3 - 20}{3 \cdot (2.715)^2}$$

$$\Rightarrow \; x_3 = 2.714$$

$$|\varepsilon_a| = \left|\frac{x_3 - x_2}{x_3}\right| \times 100 = \left|\frac{2.714 - 2.715}{2.714}\right| \times 100 = 0.009 \; < 0.05\,\%$$

the **absolute relative approximate error** is going from

$$9.45\,\% \;\Rightarrow\; 0.96\,\% \;\Rightarrow\; 0.009$$

So the **Newton-Raphson method** is **converging very fast** to the **root**, and it is **less than 5 %** and **less than 0.5 %** and **less than 0.05 %** which means that **at least three significant digits** are **correct** in the **solution** because for **one significant** to **be correct** it **needs at least 5 %,** and for **two 0.5 %** and for **three 0.05 %.** So **three significant digits** which are **at least correct** in the **root** of the **equation** can be taken.

$$x = 2.714$$

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Example

Use the Newton-Raphson iteration method to estimate the root of the following function employing an initial guess of $x_0 = 0$: $f(x) = e^{-x} - x$

Let's find the derivative of the function first, $f'(x) = \frac{df(x)}{dx} = -e^{-x} - 1$

The initial guess is $x_0 = 0$, hence,

<u>$i = 0$:</u>

$f(0) = e^{-(0)} - 0 = 1$ $\qquad f(x) = e^{-x} - x$

$f'(0) = -e^{-(0)} - 1 = -1-1=-2$ $\qquad f'(x) = \frac{df(x)}{dx} = -e^{-x} - 1$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{-2} = 0.5$$

Now $x_1 = 0.5$, hence,

<u>$i = 1$</u>

$f(0.5) = e^{-(0.5)} - (0.5) = 0.1065$ $\qquad f(x) = e^{-x} - x$

$= -e^{-(0.5)} - 1 = -1.6065$ $\qquad f'(x) = \frac{df(x)}{dx} = -e^{-x} - 1$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5 - \frac{0.1065}{-1.6065} = 0.5663$$

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Now x_2= 0.5663, hence,

i=2

$$f'(x)=\frac{df(x)}{dx}=-e^{-x}-1$$

$$f(0.5663)=e^{-(0.5663)}-(0.5663)=0.001322$$

$$=-e^{-(0.5663)}-1=-1.567622$$

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

$$x_3=x_2-\frac{f(x_2)}{f'(x_2)}=0.5663-\frac{0.001322}{-1.567622}=0.5671$$

Now x_3= 0.5671, hence,

$$f'(x)=\frac{df(x)}{dx}=-e^{-x}-1$$

$$f(0.5671)=e^{-(0.5671)}-(0.5671)=0.00006784$$

$$=-e^{-(0.5671)}-1=-1.56716784$$

Thus, the approach rapidly converges on the true root of 0.5671 to four significant digits.

i	x_i	$f(x_i)$	$f'(x_i)$	Percent $\lvert\varepsilon_r\rvert$
0	0	1	-2	---
1	0.5	0.106531	-1.6065307	100
2	0.566311003	0.001305	-1.5676155	11.709291
3	0.567143165	1.96E-07	-1.5671434	0.14672871
4	0.56714329	4.44E-15	-1.5671433	2.2106E-05
5	0.56714329	0	-1.5671433	5.0897E-13

Hence, the root is 0.5671.

H.W:

1. What are the advantages and disadvantages of Simple Iteration method and Newton-Raphson method?

2.

Use the Newton-Raphson method to estimate the root of $f(x)=e^{x}-1.5-$

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$tan^{-1}x$. Employing an initial guess of x_0= -10.

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