Line Integral in the Complex Plane.

Complex definite integrals are called (complex) **line integrals** written as:

$$\int_C f(z)dz$$

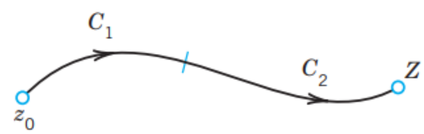
Integrated over a portion of curve C . The function z(t)=4cost + 4i sint ($-\pi \le t \le \pi$) represents the circle.

Basic Properties

1. $$\int_C [k_1 f_1(z) + k_2 f_2(z)]\,dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz.$$

2. $$\int_{z_0}^{Z} f(z)\,dz = -\int_{Z}^{z_0} f(z)\,dz.$$

3. $$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$



Example 1: find the integrals :

$$\int_0^{1+i} z^2\,dz = \frac{z^3}{3}\Big|_0^{1+i} = \frac{(1+i)^3-(0)^3}{3} = \frac{(1+i)(1+i)^2}{3} = \frac{(1+i)(1+2i+i^2)}{3} = \frac{(1+i)(2i)}{3} = \frac{-2+2i}{3}$$

$$\int_{-\pi i}^{\pi i} \cos z = \sin z \Big|_{-\pi i}^{\pi i} = (\sin \pi i) - (\sin -\pi i) = (\sin \pi i) + (\sin \pi i) = 2\sin \pi i$$

But $\left(\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})\right)$ **take** $x=\pi i$

Then : $= 2\sin \pi i = 2\,\frac{1}{2i}(e^{i\,\pi i} - e^{-i\,\pi i}) = \frac{1}{i}(e^{-\pi} - e^{\pi}) = -\frac{1}{i}(e^{\pi} - e^{-\pi}) = -2\,\frac{\sinh \pi}{i} * \frac{i}{i} =$
$2i \sinh \pi = 2i * 11.5487 = 23.0974\, i$

(but $\sinh y = \frac{(e^{y} - e^{-y})}{2}$ **, take** $y = \pi \rightarrow \sinh \pi = \frac{(e^{\pi} - e^{-\pi})}{2}$ **)**

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Cauchy's Integral Formula.

Line integral of a function generally depends not only on the endpoints but also on the choice of the path.

❖ A **simple closed path**: No intersect or touch itself

Simple

❖ A **simply connected domain** D : Every simple closed path in D encloses only points of D

Simply connected

Theorem 1 (Cauchy's Integral Formula)

If it is analytic in a simply connected domain D, then for every simple closed path:

$$\oint_C f(z)\, dz = 0.$$

A simple closed path is sometimes called a **contour** and an integral over such a path a **contour integral**.

Example 2 : **Entire Functions**

$$\oint_C e^z\, dz = 0, \qquad \oint_C \cos z\, dz = 0$$

Points Outside the Contour → f(x) Not Analytic

$\oint_C \sec z\, dz = 0,$ sec z=1/cos z is not analytic at z= ±π/2 , ±3π/2

$\oint_C \frac{dz}{z^2 + 4} = 0$ not analytic at z =±2i , outside C

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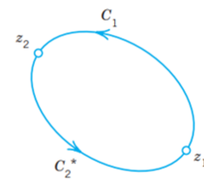
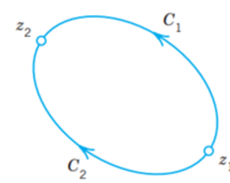
Independence of Path

The integral of f(z) is independent of path in D

$$\int_{C_1} f\,dz + \int_{C_2^*} f\,dz = 0, \qquad \text{thus} \qquad \int_{C_1} f\,dz = -\int_{C_2^*} f\,dz$$



Sign on the right disappears if we integrate in the reverse direction



$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Note : The **unit circle** mean |z|=1

دائرة الوحدة هي جميع الأعداد المركبة z التي تحقق|z|=1

Example 3 :

Integrate f (z) = exp(-z^2) counterclockwise around the **unit circle**. Indicate whether satisfy **Cauchy's integral theorem**. Show in details

Solution: exp (-z^2) is **entire تامة (analytic everywhere) inside and on** the unit circle.

الدالة تامة Entire function يعني شاملة لجميع الاعداد لانه:

- لا تحتوي مقام
- لا يوجد نقاط شاذة(Singularities)
- تحليلية في كل مكان في المستوى المركب.

نظرية كوشي تقول:

إذا كانت الدالة تحليلية تامة وعلى المسار المغلق C فإن:

$$\oint_C e^{-z^2}\,dz = 0$$ satisfy Cauchy's Integral Theorem

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Example 4 : Find singularities for the function f(z)=$\frac{1}{2z-1}$ and Check Cauchy's Theorem? 0 = نبحث عن العدد الذي يجعل المقام

Solution :

$$2z-1=0 \Rightarrow z=\frac{1}{2} \longrightarrow \left|\frac{1}{2}\right|=\frac{1}{2}<1$$ inside the unit circle.

→ Not analytic inside C →Cannot apply Cauchy's Theoremلان العدد من ضمن مجال الاعداد

Example 5: Find singularities for the function f(z)=$\frac{1}{z^4-1.1}$ and Check Cauchy's Theorem?

Solution : $z^4-1.1$=0 ايجاد العدد الذي يجعل المقام=0

Z^4=1.1 → $|z|=(1.1)^{1/4}$ ≈1.024 > 1 **outside** the unit circle.

→ No singularities قيم شاذة inside the unit circle → analytic function

→ Satisfy Cauchy's Integral Theorem

$$\oint_C \frac{1}{z^4-1.1}dz=0$$

Cauchy's integral formula

$$\oint_C \frac{f(z)}{z-z_0}dz=2\pi i f(z_0)$$ ....الصيغة العامة لايجاد قيمة التكامل وفق صيغة كوشي

Example 6 : find integrals by using **Cauchy's integral formula**?

1. $\oint \frac{e^z}{z-2}dz$

Solution :

$$\oint_C \frac{e^z}{z-2}dz=2\pi i e^z\Big|_{z=2}=2\pi i e^2=46.4268i$$

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2. $\oint \frac{z^3-6}{2z-i}dz$, solution : $\oint \frac{z^3-6}{2z-i}dz = \oint \frac{\frac{1}{2}z^3-3}{z-\frac{1}{2}i}dz = 2\pi i[\frac{1}{2}z^3-3]|_{Z=i/2}$ $=\frac{\pi}{8}-6\pi i$

3. $\oint_{|z|=2} \frac{e^z}{z}dz$

Solution : $f(z)=e^z$ (analytic everywhere) using Cauchy's Integral Formula, $z_0=0$

$\oint \frac{e^z}{z}dz = 2\pi i f(0)$, $f(0) = e^z = e^0 = 1 \rightarrow \oint \frac{e^z}{z}dz = 2\pi i$

Derivatives of Analytic Functions

الصيغ العامة لايجاد التكامل من خلال مشتقة الدالة

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

ناخذ المشتقة الاولى اذا كان المقام تربيع

$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}dz$$

وناخذ المشتقة الثانية اذا كان المقام تكعيب

Example 7 : Evaluate $\oint \frac{\cos z}{(z-\pi i)^2}dz$ and $\oint \frac{z^4-3z^2+6}{(z+i)^3}dz$ by Cauchy's integral formula?

Solution :1. $\oint \frac{\cos z}{(z-\pi i)^2}dz = 2\pi i(\cos z)'|_{Z=\pi i} = -2\pi i \sin\pi i = 2\pi i \sinh\pi$

($note$: $-\sin\pi = \sinh\pi$) من المحاضرات السابقة

2. $\oint \frac{z^4-3z^2+6}{(z+i)^3}dz = \frac{2\pi i}{2!} f''(z)|_{Z=-i}$

$f(z) = z^4-3z^2+6$, $f'(z) = 4z^3-6z$ and $f''(z)=12z^2-6$ تعوض بالمعادلة

$\oint \frac{z^4-3z^2+6}{(z+i)^3}dz = \pi i\,(z^4-3z^2+6)''|_{Z=-i} = \pi i\,(12z^2-6)\,|_{Z=-i} = \pi i\,(12(-i)^2-6) = -18\,\pi i$

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Example 8 : Evaluate $\oint \frac{e^z}{z^2}\,dz$ by Cauchy's integral formula**?**

Solution :

$\oint \frac{e^z}{z^2}\,dz = \oint \frac{e^z}{(z-0)^2}\,dz = 2\pi i\, f'(0)$ **كتابتها بالصيغة العامة**

$f(z) = e^z$, $f'(z) = e^z \rightarrow f'(0) = e^0 = 1$

$\oint \frac{e^z}{(z-0)^2}\,dz = 2\pi i\, f'(0) = 2\pi i\,(1) = 2\pi i$

HW :

1. Find the integral by using Cauchy's integral formula:

$$\oint_{|z|=1} \frac{z+1}{z-3}\,dz$$

2. Evaluate **the integrals by using the derivative (Cauchy's integral formula) ?**

$\oint \frac{z^3}{(z-2)^2}\,dz$, $\oint \frac{\ln(1+z)}{z^2}\,dz$, $\oint \frac{z^3+1}{(z-1)^3}\,dz$

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