Exponential Function

One of the most important analytic functions, the complex exponential function

$$e^z = e^{x+yi} = e^x * e^{yi} = e^x (\cos y + i \sin y)$$

$$\rightarrow e^{iy} = \cos y + i \sin y \rightarrow \textbf{Euler formula}$$

$$(e^z)' = e^z.$$

$$(e^z)' = (e^x \cos y)_x + i(e^x \sin y)_x = e^x \cos y + ie^x \sin y = e^z.$$

$$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

Example 1: Find e^z in the form u+iv and |z| if z =3+4i

Solution : $e^{x+iy} = e^x(\cos y + i \sin y)$

x=3 , y=4

$$e^{3+4i} = e^3(\cos 4 + i \sin 4) \Longrightarrow \quad e^3 \approx 20.085, \quad \cos 4 \approx -0.6536, \quad \sin 4 \approx -0.7568$$

e^{3+4i}=20.085(-0.6536+i(-0.7568))

$$e^z \approx -13.12 - 15.19i$$

$|z| = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{9+16} = 5$

Polar form in complex number , z= r(cos θ + i sin θ)=r $e^{i\theta}$

$$e^{i\theta} = \cos\theta + i \sin\theta$$

For θ=2π $\rightarrow e^{i2\pi}$=cos2π + i sin2π =1

For θ=πi/2 $\rightarrow e^{i\pi/2}$=cosπ/2 + i sinπ/2 =i

$$|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$$

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Polar Form

Example 2 : Write Z=4+3i in exponential form ($z = re^{i\theta}$):

Solution : x=4 , y=3

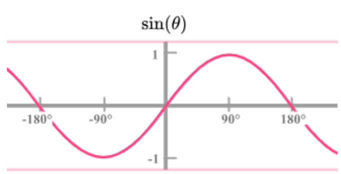
$r=|z| = \sqrt{x^2+y^2} = \sqrt{3^2+4^2} = \sqrt{9+16} = 5$

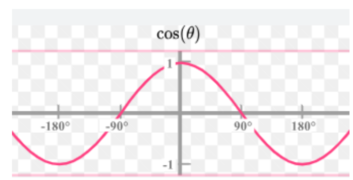
$\theta = \tan^{-1}(\frac{3}{4}) = 0.6435$ → $z = 5e^{i0.6435}$

Example 3 : Write Z=-6.3 in exponential form ($z = re^{i\theta}$):

Solution : $z=-6.3+0i$, $r=|z| = \sqrt{x^2+y^2} = \sqrt{(-6.3)^2+0^2} = \sqrt{-6.3^2} = -6.3$

$\theta = \pi$ why? , $z = -6.3e^{i\pi}$





Trigonometric Function

Just as extended the real to the complex, now extend the real trigonometric functions to complex trigonometric functions, can do this by the use of the **Euler formulas**

$e^{ix} = \cos x + i \sin x$ ……(1) , $e^{-ix} = \cos x - i \sin x$ ……(2)

$e^{ix} + e^{-ix} = (\cos x + i \sin x) + (\cos x - i \sin x) = 2\cos x$ → $\mathbf{\cos x = \frac{1}{2}(e^{ix} + e^{-ix})}$

also take $e^{ix} - e^{-ix}$ → $\mathbf{\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})}$

In Z- form: $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$ ….(3)

Hyperbolic Function

The complex hyperbolic cosine and sine are defined by the formulas

$$\cosh y = \frac{e^y + e^{-y}}{2} \qquad \sinh y = \frac{e^y - e^{-y}}{2} \qquad \ldots\ldots\ldots\ldots(4)$$

$$(\cosh z)' = \sinh z, \qquad (\sinh z)' = \cosh z,$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z},$$

$$\operatorname{sech} z = \frac{1}{\cosh z}, \qquad \operatorname{csch} z = \frac{1}{\sinh z}.$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$$

$$\cosh^2 z - \sinh^2 z = 1, \quad \cosh^2 z + \sinh^2 z = \cosh 2z$$

Example 4: prove that $\cos z = \cos x \cosh y - i \sin x \sinh y$

Solution : z=x+iy:

From eq. (3):$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$, When z=x+iy →

cos z=$\frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)})$=$\frac{1}{2}(e^{ix}e^{-y} + e^{-ix}e^{y})$ From equations (1),(2) we get :

cosz =$\frac{1}{2}\left((\cos x + i \sin x)e^{-y} + (\cos x - i \sin x)e^{y}\right)$=$\frac{1}{2}(\cos x\, e^{-y} + i \sin x\, e^{-y} + \cos x\, e^{y} - i \sin x\, e^{y})$

cosz =$\cos x \frac{(e^{-y} + e^{y})}{2} + i \sin x \frac{(e^{-y} - e^{y})}{2} = \cos x \frac{(e^{-y} + e^{y})}{2} - i \sin x \frac{(e^{y} - e^{-y})}{2}$

From eq. (4):

$$\cosh y = \frac{e^{y} + e^{-y}}{2} \qquad \sinh y = \frac{e^{y} - e^{-y}}{2} \quad \Rightarrow \quad \cos z = \cos x \cosh y - i \sin x \sinh y$$

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Logarithm

Complex logarithm, which is more complicated than the real logarithm, The **natural logarithm** of z=x+iy is denoted by ln(z) , defined as the inverse of the exponential function: w=ln z

$e^w = z = re^{i\theta}$ but w =u+ iv

$e^w = e^{u+iv} = re^{i\theta}$

$z=re^{i\theta}$, take ln , $\ln z = \ln(re^{i\theta}) = \ln r + \ln(e^{i\theta}) = \ln r + i\theta$ (note: r=|z|)

→ $\ln z = \ln|z| + i \,\mathrm{Arg}\, z$

$\ln(z_1 z_2) = \ln z_1 + \ln z_2$, $\ln(z_1/z_2) = \ln z_1 - \ln z_2$

Example 5: Find Ln z when Z=5+3i

Solution:

$$\ln z = \ln|z| + i\arg(z) \qquad |z| = \sqrt{5^2+3^2} = \sqrt{25+9} = \sqrt{34}$$

$$\arg(z) = \tan^{-1}\left(\frac{3}{5}\right) \qquad \arg(z) = \tan^{-1}(0.6)$$

$$\arg(z) \approx 0.5404 \text{ radians}$$

$$\ln z = \ln(\sqrt{34}) + i(0.5404)$$

$$\boxed{\ln(5+3i) \approx 1.763 + 0.5404\,i}$$

Example 6 : Find value of Ln(4+3i) and graph in the complex plane

$$\ln z = \ln|z| + i(\arg z + 2\pi k) \qquad |z| = \sqrt{4^2+3^2} = \sqrt{16+9} = 5$$

$$\theta = \arg(z) = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\ln(4+3i) = 1.6094 + 0.6435\,i \qquad \theta \approx 0.6435 \text{ radians}$$

Lie on a vertical line at x=1.6094

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