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Trigonometric Integrals تكاملات الدوال المثلثية

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. تتضمن التكاملات المثلثية علاقات جبرية للدوال المثلثية الأساسية الست

$$\int \sec^2 x\,dx = \tan x + C.$$

$\cos^2x+\sin^2x=1$, $\cos 2x=\cos^2x-\sin^2x$, $\sin^2 x = \frac{1-\cos 2x}{2}$ (حفظ)

Example 1 Evaluate:

$$\int \sin^3 x \cos^2 x\,dx.$$

Solution :

$$\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x \sin x\,dx$$

$$= \int (1-\cos^2 x)\cos^2 x \sin x\,dx = \int (\cos^2 x - \cos^4 x)\sin x\,dx = \int (\cos^2 x - \cos^4 x)\sin x\,dx$$

$$= \frac{-1}{-1}\int \cos^2 \sin x\,dx - \frac{-1}{-1}\int \cos^4 x\ \sin x\,dx = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$$

Example 2 Evaluate

$$\int \cos^5 x\,dx.$$

Solution :

$$= \int \cos^4 x \cos x\,dx$$

$$= \int (\cos^2 x)^2 \cos x\,dx$$

$$= \int (1-\sin^2 x)^2 \cos x\,dx = \int (1 - 2\sin^2 x + \sin^4 x)\cos x\,dx$$

$$= \int \cos x\,dx - \int 2\sin^2 x \cos x\,dx + \int \sin^4 x \cos x\,dx = \sin x - 2\frac{\sin^3 x}{3} + \frac{\sin^5 x}{5} + c$$

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Trigonometric Substitutions تعويضات الدوال المثلثية

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. يتم التعويض عن المتغير في التكامل بدالة مثلثية معينة حسب شكل التكامل

Example 3 : Evaluate

$$\int \frac{dx}{\sqrt{4+x^2}}$$

Solution: substitute x = 2 tan u

dx= $2\, sec^2 u\, du$

الفرضية من اجل تحويل شكل $\sqrt{4+x^2}$ بالشكل $\sqrt{1+tan^2 u}$ لان $1+tan^2 u = sec^2 u$

وبالتالي $secu = \sqrt{sec^2 u}$

$$\int \frac{2sec^2u\, du}{\sqrt{4+(2tan\, u)^2}} = \int \frac{2sec^2u\, du}{\sqrt{4+4\, tan^2 u}} = \int \frac{2sec^2u\, du}{\sqrt{4(1+tan^2 u)}} \int \frac{2sec^2u\, du}{2\sqrt{(1+tan^2 u)}}$$

$$\int \frac{2sec^2u\, du}{2\sqrt{sec^2 u}} = \int \frac{sec^2u\, du}{sec\, u} = \int sec\, u\, du = \ ln\, \left| sec\, u\ +\ tan\, u \right|\ +\ C$$

Example 4: Evaluate

$$\int \frac{dx}{1+x^2}$$

Solution :

Let x= $tan\theta$ dx= $sec^2\theta\, d\theta$ $\rightarrow \int \frac{sec^2\theta\, d\theta}{1+tan^2\theta} = \int \frac{sec^2\theta\, d\theta}{sec^2\theta} = \int d\theta = \theta$

But x= $tan\theta$ $\rightarrow \theta = tan^{-1}x$ then

$$\int \frac{dx}{1+x^2} = tan^{-1}x$$

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Example 5: calculate $\int \frac{dx}{\sqrt{1-x^2}}$

Solution : substitute x= $sin\theta$ dx= $cos\theta \rightarrow \int \frac{dx}{\sqrt{1-x^2}} = \int \frac{cos\theta}{\sqrt{1-sin^2\theta}} = \int \frac{cos\theta\, d\theta}{\sqrt{cos^2\theta}} =$

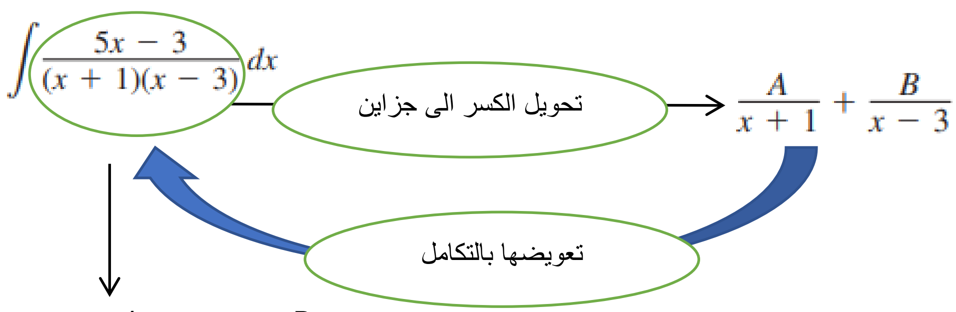
$\int \frac{cos\theta\, d\theta}{cos\theta} = \int d\theta = \theta$ but x= $sin\theta \rightarrow \theta = sin^{-1}x$

Then $\int \frac{dx}{\sqrt{1-x^2}} = sin^{-1}x$

Integration of Rational Functions by Partial Fractions

تكامل الدوال الكسرية بالكسور الجزئية

Express a rational function (a quotient of polynomials) as sum of simpler fractions, called **partial fractions**



$$= \int \frac{A}{x+1}dx + \int \frac{B}{x-3}\,dx$$

A , B حساب قيم

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-3)} \rightarrow \frac{5x-3}{(x+1)(x-3)} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)} = \frac{Ax-3A+Bx+B}{(x+1)(x-3)}$$

Then $5x-3 = Ax-3A+Bx+B = Ax+Bx-3A+B$

المعاملات متساوية للمتغير المرفوع لنفس القوة يعني $5x = Ax + Bx$ والثوابت في كلا الطرفان متساوية $-3 = -3A + B$

$A + B = 5, \quad -3A + B = -3 \longrightarrow A = 2 \text{ and } B = 3$

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The Definite integral التكامل المحدد

$$\int_a^b f(x)\,dx.$$

Example 6 : find the integrals

1. $\int_3^1 7dx$ Solution : $= 7x\,\Big|_3^1 = 7*(1-3) = 7*(-2) = -14$

2. $\int_1^2 3u^2\ du$ Solution : $= 3\,\frac{u^3}{3}\,\Big|_1^2 = 3*\left[\frac{2^3}{3} - \frac{1^3}{3}\right] = 3*\frac{8-1}{3} = 7$

The basis for formulating definite integrals is the construction of approximations by finite sums. Three examples of this process:

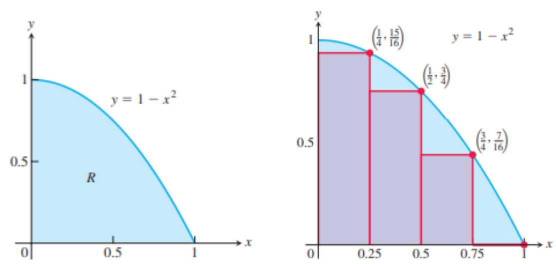
التكاملات المحددة تبني تقريبات باستخدام المجاميع المحدودة، ثلاث امثلة لهذه العملية:

- ❖ Finding the area under a graph.حساب المساحة تحت المنحنى او الرسم
- ❖ The distance traveled by a moving object. المسافة التي يتحركها الجسم
- ❖ The average value of a function معدل قيمة الدالة

Definite Integral applications تطبيقات التكامل المحدد

Area under the curve مساحة تحت المنحنى

Included find the area of the shaded region R that lies above the x-axis, below the graph of $y = 1 - x^2$, and between the vertical lines x = 0 and x = 1



$$A \approx \frac{15}{16}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + \frac{7}{16}\cdot\frac{1}{4} + 0\cdot\frac{1}{4} = \frac{17}{32} = 0.53125$$

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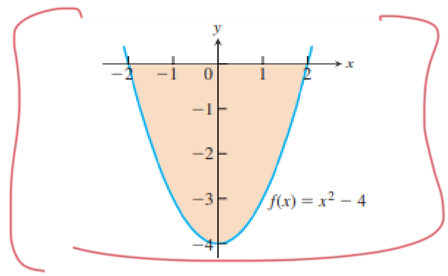
Example 7 : Compute the area between the graph and the x-axis over [-2, 2] , the function is $f(x) = x^2 - 4$

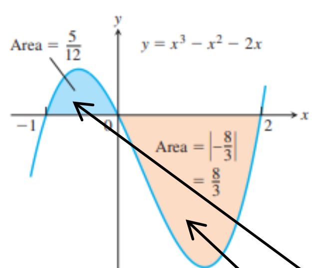


Solution : $$\int_{-2}^{2} f(x)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right) = -\frac{32}{3}$$

المساحة تكون بالموجب حتى وان ظهرت القيمة بالسالب. يعني المساحة= $\frac{32}{3}$

Example 8: Find the total area of the region between the x-axis and the graph of

$f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$



Solution :

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

نجمع المساحتين لتعطي المساحة الكلية مع جعل القيم السالبة موجبة باستخدام القيمة المطلقة

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$

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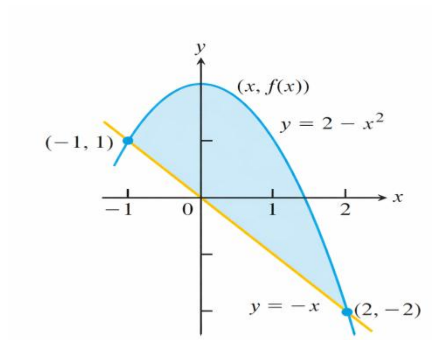
Areas Between Curves المساحة مابين المنحنيات

Example 9 :

Find the area of the region enclosed by the parabola القطع المكافئ $y_1 = 2 - x^2$ and the line $y_2 = -x$



Solution :

$$2 - x^2 = -x$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1, \quad x = 2.$$

نساوي الدالتين لمعرفة نقاط التقاطع مابين الخط المستقيم والمنحنى من اجل معرفة نقطة البداية والنهاية للتكامل

Area $= \int_{-1}^{2} (y_1 - y_2)\, dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\, dx$

Area $= \int_{-1}^{2} [2 - x^2 + x] dx = (2x - \frac{x^3}{3} + \frac{x^2}{2}) \Big| _{-1}^{2} = [2*(2) - \frac{(2)^3}{3} + \frac{(2)^2}{2}] - \left[2 * (-1) - \frac{(-1)^3}{3} + \frac{(-1)^2}{2}\right]$

$$= \left[4 - \frac{8}{3} + \frac{4}{2}\right] - \left[-2 - \frac{-1}{3} + \frac{1}{2}\right]$$

$$= \frac{24-16+12}{6} - \frac{-12-(-2)+3}{6}$$

$$= \frac{20}{6} - \frac{-7}{6} = \frac{27}{6} = \frac{9}{2}$$

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