Complex Function

Complex analysis is concerned with complex functions that are differentiable in some domain. Hence we should first say what we mean by a complex function and then define the concepts of limit and derivative in complex. This discussion will be similar to that in calculus.

يهتم التحليل المركب بالدوال المركبة القابلة للتفاضل. لذا ينبغي:

تحديد مفهوم الدالة المركبة.

S = set of complex numbers , f (z)= function = value of f at z on S

تعريف مفهومي النهاية والمشتقة في الدوال المركبة, بشكل مشابه لما هو عليه في حساب التفاضل والتكامل.

Derivative

The derivative of a differentiable complex function f at z_o , and $\Delta z=z-z_o$, we have:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

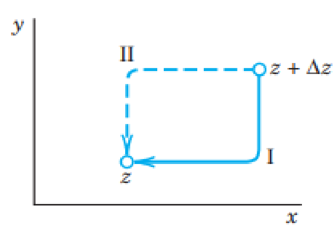
The function $f(z)= z^2$ is differentiable for all z and has the derivative f '(z)=2z because

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\,\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z.$$

$\bar{z} = x - iy$ **المرافق المركّب**

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\overline{(z + \Delta z)} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y}.$$

If Δy=0 this is +1. If Δx=0 , this is -1,then the limit as Δz →0 does not exist at any z.



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Analytic Function

Complex analysis is concerned with the theory and application of "analytic functions," that is, functions that are differentiable in some domain, so that we can do "calculus in complex." The definition is as follows.

التحليل المركب يستخدم الدوال التحليلية القابلة للتفاضل ، بحيث يمكننا إجراء "حساب التفاضل والتكامل في المركب". التعريف كما يلي:

A function is analytic in a domain D if f(z) is differentiable at all points of D.

تُسمى الدالة تحليلية في مجال D إذا كانت f(z) قابلة للاشتقاق عند جميع نقاط D.

Example 1: Find the value of the derivative of the functions f(z)= $(z-4i)^8$ at z=3+4i ,

f(z)=$\frac{z-i}{z+i}$ at z=i

2. f(z)= $(z-4i)^8$ at z=3+4i

solution

$$f'(z) = 8(z-4i)^7$$

$$z - 4i = (3+4i) - 4i = 3$$

$$f'(3+4i) = 8(3)^7$$

$$f'(3+4i) = 8 \times 2187 = 17496$$

1. f(z)=$\frac{z-i}{z+i}$ at z=i

solution

$$f'(z) = \frac{(z+i)\cdot 1 - (z-i)\cdot 1}{(z+i)^2}$$

$$f'(z) = \frac{2i}{(z+i)^2}$$

$$f'(i) = \frac{2i}{(i+i)^2}$$

$$f'(i) = \frac{2i}{(i+i)^2}$$

$$(i+i)^2 = (2i)^2 = -4$$

$$f'(i) = \frac{2i}{-4} = -\frac{i}{2}$$

Cauchy–Riemann Equations معادلات كوشي-ريمان

Complex analysis require that function to be differentiable in that domain. The Cauchy–Riemann equations are the most important equations, provide a criterion (a test) for the analyticity of a complex function

يتطلب التحليل المركب أن تكون الدالة قابلة للتفاضل في ذلك المجال. وتُعد معادلات كوشي-ريمان أهم المعادلات، إذ تُوفر معيارًا (اختبارًا) لتحليلية الدالة المركبة.

$$w = f(z) = u(x, y) + iv(x, y).$$

Roughly, *f* is analytic in a domain *D* if and only if the first partial derivatives of *u* and v satisfy the two **Cauchy–Riemann equations**

تكون الدالة f تحليلية في المجال D إذا كانت المشتقات الجزئية الأولى لـ u و v تحقق معادلتي كوشي-ريمان.

$$u_x = v_y, \qquad u_y = -v_x$$

$$u_x=\frac{du}{dx} \ , \ u_y=\frac{du}{dy} \qquad \text{or} \qquad v_x=\frac{dv}{dx} \ , \ v_y=\frac{dv}{dy}$$

Example 2: prove that $z=x^2-y^2+2xyi$ satisfy the two Cauchy–Riemann equations ($u_x=v_y$ and $u_y= -v_x$)

Solution : From the equation: $z=x^2-y^2+2xyi \rightarrow u=x^2-y^2$, $v=2xy$

For $u_x=v_y$ $\frac{du}{dx}=\frac{dv}{dy}$ 2x=2x satisfy $u_x=v_y$	For $u_y=-v_x$ $\frac{du}{dy}=-\frac{dv}{dx}$ -2y=-2y satisfy $u_y=-v_x$

Then satisfy Cauchy–Riemann equations

Example 3: if f(z) $=z^2$ is analytic for all *z* and satisfy Cauchy–Riemann equations , is $\bar{z}$=x-iy analytic for all z? (Note: is $\bar{z}=a-bi$ conjugate for $z=a+bi$)

Solution : u=x and v=-y

For $u_x=v_y$: $\frac{du}{dx}=\frac{dv}{dy} \rightarrow$ 1=-1 then not analytic.

Example 4 : Is f(z)= u(x, y)+iv(x, y)= e^x(cosy + i siny) analytic ?

Solution. We have $u = e^x \cos y, v = e^x \sin y$ and by differentiation

$$u_x = e^x \cos y, \qquad v_y = e^x \cos y$$
$$u_y = -e^x \sin y, \qquad v_x = e^x \sin y.$$

Then satisfy Cauchy–Riemann equations and conclude that *f*(*z*) is analytic for all *z*.

Laplace's Equation معادلة لابلاس

The great importance of complex analysis in engineering mathematics results mainly from the fact that both real part and imaginary part of analytic function satisfy Laplace's equation.

أهمية التحليل المركب في الرياضيات الهندسية يكون من خلال تحقيق الجزء الحقيقي والجزء الخيالي للدالة التحليلية لمعادلة لابلاس.

الدالة التوافقية Harmonic Function: تمتلك مشتقات جزئية من الرتبة الثانية مستمرة وتحقق معادلة لابلاس

Laplace's Equation

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then both u and v satisfy **Laplace's equation**

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

(∇^2 read "nabla squared") *and*

$$\nabla^2 v = v_{xx} + v_{yy} = 0.$$

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Example 5: is the function $f(z)=z^2$ satisfy Laplace's equation ?

Solution :

$Z=x+iy$, substitute in function f(z):

$f(z)=(x+iy)^2=x^2-y^2+i(2xy) \rightarrow u=x^2-y^2$, then $u_x=2x$ and $u_{xx}=2$

$u_y=-2y$ and $u_{yy}=-2$

and $v=2xy$, then $v_x=2y$ and $v_{xx}=0$

$v_y=2x$ and $v_{yy}=0$

Check Laplace's equations:

$\nabla^2 u= u_{xx}+ u_{yy}=2+(-2)=0$

$\nabla^2 v= v_{xx}+ v_{yy}= 0+0=0 \rightarrow$ both satisfy Laplace's equation

Example 6 : is the function $f(z)=e^x (\cos y+i \sin y)$ satisfy Laplace's equation?

Solution : $f(z)=e^x\cos y+i\, e^x \sin y$

$u= e^x \cos y$: $u_x=e^x \cos y$ and $u_{xx}= e^x \cos y$

$u_y=- e^x \sin y$ and $u_{yy} = -e^x \cos y$

$v= e^x \sin y$: $v_x= e^x \sin y$ and $v_{xx}= e^x \sin y$

$v_y= e^x \cos y$ and $v_{yy}=- e^x \sin y$

Check Laplace's equations:

$\nabla^2 u= u_{xx}+ u_{yy}= e^x \cos y + (- e^x \cos y)=0$ and $\nabla^2 v= v_{xx}+ v_{yy}= e^x \sin y+(- e^x \sin y)=0$

Satisfy Laplace's equation

Example 7: is the function $f(z)=x^2-y^2+i(2xy+y)$ satisfy Cauchy–Riemann equations and Laplace's equation?

Solution :

$u= x^2-y^2$ and $v=2xy+y$

1. Check Cauchy–Riemann equations:

$u_x=\frac{du}{dx}=2x$, $u_y=\frac{du}{dy}=-2y$ or $v_x=\frac{dv}{dx}=2y$, $v_y=\frac{dv}{dy}=2x+1$

Then $u_x \neq v_y$ → Cauchy–Riemann equations are NOT satisfied

2. Check Laplace's equation:

$u_x=2x$ and $u_{xx}= 2$
$u_y=- 2y$ and $u_{yy} = -2$
$v_x= 2y$ and $v_{xx}= 0$
$v_y= 2x+1$ and $v_{yy}= 0$

Then $\nabla^2u= u_{xx}+ u_{yy}= 2 + -2=0$ and $\nabla^2v= v_{xx}+ v_{yy}= 0+ 0= 0$

Satisfy Laplace's equation

HW find the derivative of the functions

1. $(iz^3+3z^2)^3$ at 2i
2. $z^3/(z+i)^3$ at i

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