



## Differentiation

### Definition of derivative:

The derivative is the rate of change of a function with respect to its variable, or the slope of the tangent line to the curve at a given point.

المشتقة هي معدل تغير دالة بالنسبة لمتغيرها، أو هي ميل المماس للمنحنى عند نقطة معينة.

**DEFINITION** The **derivative** of the function  $f(x)$  with respect to the variable  $x$  is the function  $f'$  whose value at  $x$  is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h},$$

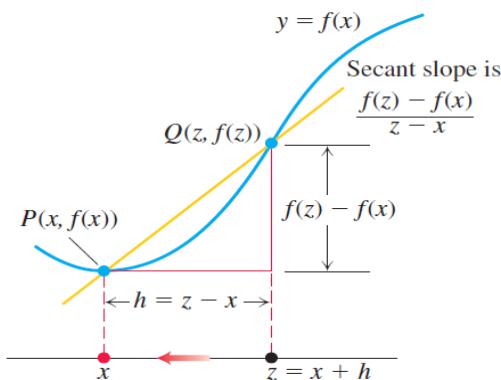
provided the limit exists.

صيغة بديلة:

### Alternative Formula for the Derivative

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

If we write  $z = x + h$ , then  $h = z - x$  and  $h$  approaches 0 if and only if  $z$  approaches  $x$ .  
تقرب  $h$  من 0 إذا وفقط إذا اقتربت  $z$  من  $x$ .





**Example 1:** Find the slope of the curve  $y = 1/x$  at any point  $x = a \neq 0$ . What is the slope at the point  $x = -1$ ?

Solution:

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{a + h} - \frac{1}{a}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \frac{a - (a + h)}{a(a + h)} \\ &= \lim_{h \rightarrow 0} \frac{-h}{ha(a + h)} = \lim_{h \rightarrow 0} \frac{-1}{a(a + h)} = -\frac{1}{a^2}.\end{aligned}$$

**Example 2 :** Applying the Definition, differentiate  $f(x) = \frac{x}{x-1}$

**Solution** We use the definition of derivative, which requires us to calculate  $f(x + h)$  and then subtract  $f(x)$  to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x-1} \quad \text{and} \quad f(x + h) = \frac{(x + h)}{(x + h) - 1}, \text{ so}$$

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x + h}{x + h - 1} - \frac{x}{x - 1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{(x + h)(x - 1) - x(x + h - 1)}{(x + h - 1)(x - 1)} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{-h}{(x + h - 1)(x - 1)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(x + h - 1)(x - 1)} = \frac{-1}{(x - 1)^2}.\end{aligned}$$



### Example 3:

(a) Find the derivative of  $f(x) = \sqrt{x}$  for  $x > 0$ .  
(b) Find the tangent line to the curve  $y = \sqrt{x}$  at  $x = 4$ .

#### Solution

(a) We use the alternative formula to calculate  $f'$ :

$$\begin{aligned}f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} \\&= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\&= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \quad \frac{1}{a^2 - b^2} = \frac{1}{(a - b)(a + b)} \\&= \lim_{z \rightarrow x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \quad \text{Cancel and evaluate.}\end{aligned}$$

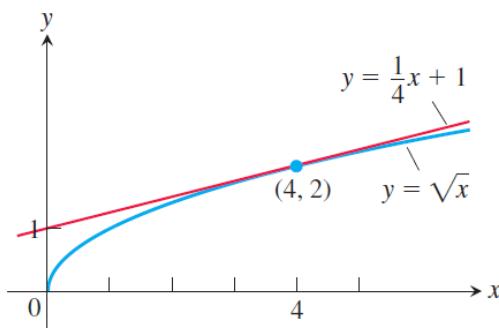
(b) The slope of the curve at  $x = 4$  is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point  $(4, 2)$  with slope  $1/4$  (Figure 3.5):

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1.$$





## Differentiation Rules:

If  $f(x)$  or  $y$  is differentiable function of  $X$  and  $C$  is constant then:

$$1- \frac{d}{dx} (c) = 0$$

$$2- \frac{d}{dx} (c f(x)) = c \frac{d f(x)}{dx}$$

$$3- \frac{d}{dx} (f(x) \mp g(x)) = \frac{d f(x)}{dx} \mp \frac{d g(x)}{dx} = f'(x) \mp g'(x)$$

$$4- \frac{d}{dx} (f(x) g(x)) = f(x) g'(x) + g(x) f'(x)$$

$$5- \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$6- \frac{d}{dx} (x^n) = n x^{n-1}$$

$$7- \frac{d}{dx} (f(x))^n = n (f(x))^{n-1} \cdot f'(x)$$

### Example 4:

Differentiate the following powers of  $x$ :

(a)  $x^3$       (b)  $x^{2/3}$       (c)  $x^{\sqrt{2}}$       (d)  $\frac{1}{x^4}$

### Solution

$$(a) \frac{d}{dx} (x^3) = 3x^{3-1} = 3x^2$$

$$(b) \frac{d}{dx} (x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$$

$$(c) \frac{d}{dx} (x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1}$$

$$(d) \frac{d}{dx} \left( \frac{1}{x^4} \right) = \frac{d}{dx} (x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$



**Example 5:** Find the derivative of the functions :

1-

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \\ &= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5 \end{aligned}$$

2-  $f(x) = x^7(x^5 + x^2)$

$$f'(x) = x^7(5x^4 + 2x) + 7x^6(x^5 + x^2)$$

3-  $f(x) = \frac{5}{x}$  ,  $f(x) = 5x^{-1}$  ,  $f'(x) = -5x^{-2} = -\frac{5}{x^2}$

4-  $f(x) = \frac{3x^2 - 5x}{7}$  ,  $f'(x) = \frac{1}{7}(6x - 5)$



## Second- and Higher-Order Derivatives

If  $y = f(x)$  is a differentiable function, then its derivative  $f'(x)$  is also a function. If  $f'$  is also differentiable, then we can differentiate  $f'$  to get a new function of  $x$  denoted by  $f''$ . So  $f'' = (f')'$ . The function  $f''$  is called the **second derivative** of  $f$  because it is the derivative of the first derivative.

إذا كانت الدالة  $y = f(x)$  قابلة للاشتاقاق، فإن مشتقتها  $f'(x)$  هي أيضاً دالة. وإذا كانت  $f'$  قابلة للاشتاقاق أيضاً، فيمكننا اشتقاق  $f''$  للحصول على دالة جديدة  $f''$  نرمز لها بـ  $f'' = (f')'$ . إذن  $f''$  تسمى الدالة  $f''$  بالمشتقة الثانية للدالة  $f$  لأنها مشتقة المشتقة الأولى.

It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

If  $y = x^6$ , then  $y' = 6x^5$  and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx} (6x^5) = 30x^4.$$

If  $y''$  is differentiable, its derivative,  $y''' = dy''/dx = d^3y/dx^3$ , is the **third derivative** of  $y$  with respect to  $x$ .

$$y^{(n)} = \frac{d}{dx} y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y$$

The names continue as you imagine, with

denoting the  **$n$ th derivative** of  $y$  with respect to  $x$  for any positive integer  $n$ .



### Example 6:

The first four derivatives of  $y = x^3 - 3x^2 + 2$  are

$$\text{First derivative: } y' = 3x^2 - 6x$$

$$\text{Second derivative: } y'' = 6x - 6$$

$$\text{Third derivative: } y''' = 6$$

$$\text{Fourth derivative: } y^{(4)} = 0.$$

## Derivatives of Trigonometric Functions

Because  $\sin x$  and  $\cos x$  are differentiable functions of  $x$ , the related functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \csc x = \frac{1}{\sin x}$$

$$1- \frac{d}{dx} (\sin u) = \cos u \cdot \frac{du}{dx}$$

$$2- \frac{d}{dx} (\cos u) = -\sin u \cdot \frac{du}{dx}$$

$$3- \frac{d}{dx} (\tan u) = \sec^2 u \cdot \frac{du}{dx}$$

$$4- \frac{d}{dx} (\cot u) = -\csc^2 u \cdot \frac{du}{dx}$$

$$5- \frac{d}{dx} (\sec u) = \sec u \tan u \cdot \frac{du}{dx}$$

$$6- \frac{d}{dx} (\csc u) = -\csc u \cot u \cdot \frac{du}{dx}$$



**Example 7 :**

a)  $y = x^2 \sin x$ : 
$$\begin{aligned} \frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + 2x \sin x \\ &= x^2 \cos x + 2x \sin x. \end{aligned}$$

b)  $y = \frac{\cos x}{1 - \sin x}$ :

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1 - \sin x) \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} && \text{Quotient Rule} \\ &= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2} \\ &= \frac{1 - \sin x}{(1 - \sin x)^2} && \sin^2 x + \cos^2 x = 1 \\ &= \frac{1}{1 - \sin x} \end{aligned}$$

c) Find  $y''$  if  $y = \sec x$ .

$$\begin{aligned} y &= \sec x \\ y' &= \sec x \tan x \\ y'' &= \frac{d}{dx}(\sec x \tan x) \\ &= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \\ &= \sec x (\sec^2 x) + \tan x (\sec x \tan x) \\ &= \sec^3 x + \sec x \tan^2 x \end{aligned}$$



d)  $y = \sin(x^5 - x)$

$$y' = [\cos(x^5 - x)] \cdot (5x^4 - 1)$$

e)  $y = \cos\left(\frac{x}{x+2}\right)$

$$y' = -\sin\left(\frac{x}{x+2}\right) \left(\frac{(x+2)-x}{(x+2)^2}\right) = -\left(\frac{2}{(x+2)^2}\right) \sin\left(\frac{x}{x+2}\right)$$

f)  $y = \sin^5 x = (\sin x)^5$

$$y' = 5(\sin x)^4 \cdot \cos x = 5 \cos x \sin^4 x$$



### EXERCISES:

**Q1/ Using the definition, calculate the derivatives of the functions. Then find the values of the derivatives as specified.**

1.  $f(x) = 4 - x^2$  ;  $f'(-3), f'(0)$

2.  $f(x) = \frac{1-x}{2x}$  ;  $f'(-1), f'(1)$

**Q2/ Find the derivatives of the functions**

1.  $y = (x-1)(x+2)$

2.  $y = \frac{5x+1}{2\sqrt{x}}$

3.  $y = \frac{3}{x} + 5 \sin x$

4.  $y = \frac{x+3 \cot x}{5-2 \csc x}$

5.  $y = \tan(5 - \sin 2x)$

6.  $y = \tan^5 x^4$

**Q3/**

1. Find  $y''$  if  $y = \frac{4x^3}{3} - x$

2. Find  $\frac{d^4y}{dx^4}$  if  $y = x^6 - 2x^4 + 3 \cos x$