Vectors

Important properties of linear systems can be described with the concept and notation of vectors. يمكن وصف الخصائص المهمة للأنظمة الخطية باستخدام مفهوم المتجهات ورموزها

A vector: quantity has magnitude and direction. كمية تمتلك قيمة واتجاه

Scalar : is a quantity that has magnitude alone (mass, time, or speed) كمية تمتلك قيمة

Zero vector: Vector has entries = 0 , no length and no direction.

$$\text{Zero vector} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The concept is connects vectors to ordinary systems of equations.

ربط المتجهات بأنظمة المعادلات الاعتيادية

The term vector appears in variety of mathematical and physical contexts, "vector spaces". يظهر مصطلح "المتجه" في سياقات رياضية وفيزيائية متنوعة "فضاءات المتجه"

Vectors in R (R^2 , R^3, R^n)

A matrix with only one column is called a **column vector**. For example of vectors are (u ,v, w) with two entries(x_1, x_2) مدخلان:

$u=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $v=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $w=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **column vector** , $\begin{bmatrix} 1 & 4 & 2 \end{bmatrix}$ **raw vector**

Where: x and y= real numbers, R^2=vector contains 2 entries,

Two vectors in R^2 are **equal** if and only if their **corresponding entries** are **equal**.

$$\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} \quad , \quad \begin{bmatrix} 4 \\ 7 \end{bmatrix} \neq \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

R^3=vector contains 3 entries. $\mathbf{u}=\begin{bmatrix} \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{bmatrix}$

R^n=vector contains n entries. $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

Geometric Descriptions of R^2

Consider a rectangular coordinate الاحداثيات المتعامدة system in the plane, each point determined by pair of numbers (a , b) with the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$, and regard R^2 as set of all points.

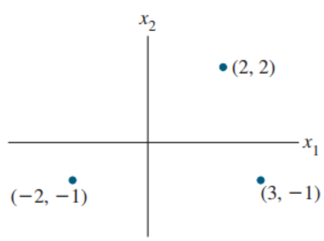


FIGURE 1 Vectors as points.

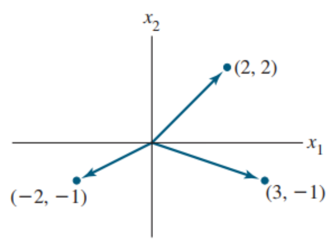


FIGURE 2 Vectors with arrows.

Operations on vectors (addition, subtraction, scalar multiplication)

Algebraic Properties of R^n

For all u; v; w in R^n and scalars قيم عددية c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

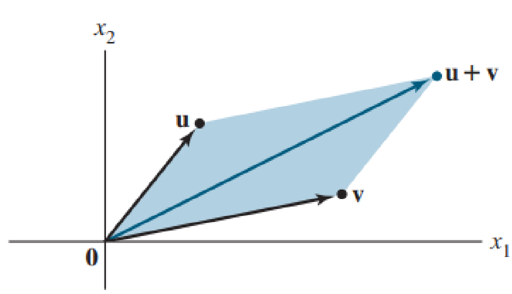
$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

$(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

$\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Parallelogram Rule for Addition

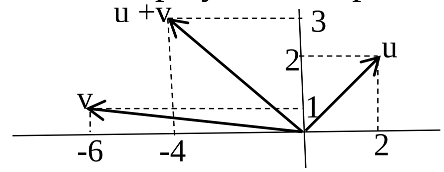
If u and v in R^2 are represented as points in the plane, then u + v as shown in figure.



Example 1 : Two vectors u = $\begin{bmatrix}2\\2\end{bmatrix}$ and v = $\begin{bmatrix}-6\\1\end{bmatrix}$, find u+v and display it in the plane ?

Solution :

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}-4\\3\end{bmatrix}$$



Example 2: Let **u** = $\begin{bmatrix}3\\-1\end{bmatrix}$, display the vectors **u**, 2**u**, and -$\frac{2}{3}$**u** on a graph?

Solution : **u** = $\begin{bmatrix}3\\-1\end{bmatrix}$, **2u** = $\begin{bmatrix}6\\-2\end{bmatrix}$, -$\frac{\mathbf{2}}{\mathbf{3}}$**u** = $\begin{bmatrix}-2\\\frac{2}{3}\end{bmatrix}$

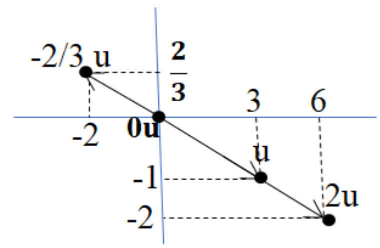


Figure 3 Typical multiples of u

Example 3 : Two vectors **u** = $\begin{bmatrix}1\\-2\end{bmatrix}$ and **v** = $\begin{bmatrix}\mathbf{2}\\\mathbf{5}\end{bmatrix}$ in R^2, find the **subtraction** (u-v) ?

Solution :

$$u-v=\begin{bmatrix}1-2\\-2-5\end{bmatrix}\rightarrow \qquad u-v=\begin{bmatrix}-1\\-7\end{bmatrix}$$

ملاحظة: في بعض الاحيان يكتب المتجه $\begin{bmatrix}3\\-1\end{bmatrix}$ بالشكل (3 , -1) ويجب ان نميزه عن مصفوفة بصف وعمودين [3 -1]

$$\begin{bmatrix}3\\-1\end{bmatrix}\neq\begin{bmatrix}3 & -1\end{bmatrix}$$

The Length of a Vector

If **v** is in R^n, with entries v_1 , …., v_n, then **Length** of **v** = $\sqrt{\mathbf{v.v}}$ is the nonnegative scalar and equal to :

$$||\mathbf{v}||=\sqrt{v.v}=\sqrt{v_1^2+v_2^2+\cdots+v_n^2}\text{.........(1)}$$

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Unit Vector has a magnitude of 1 i.e. ||u||=1

Can be creating u from v by: u= v **.** 1/||v||

Example 4: Let **v** = (1, - 2, 2, 0) , Find a unit vector **u** in the same direction as **v**?

Solution : First, compute the length of **v**:

$$\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v} = (1)^2 + (-2)^2 + (2)^2 + (0)^2 = 9$$
$$\|\mathbf{v}\| = \sqrt{9} = 3$$

Then, multiply **v** by 1/||v|| to obtain u: $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{3}\mathbf{v} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$

To check that ||u||=1: $\|u\| = \sqrt{(1/3)^2 + (-\frac{2}{3})^2 + (\frac{2}{3})^2 + 0^2} = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9} + 0} = 1$

Dot product and Cross product

Dot product

The equation is : Dot product **=** **u.v** **=** $\mathbf{u}^T$ **v**= ||**u**|| ||**v**|| **cos θ** **……. (2)**

= scalar value without brackets

Where :

u and v are vectors in R^n

Dot product is inner product of u and v

$\mathbf{u}^T$ called the transpose of u , **transpose column vector (u)** to **row vector**.

For example :

$$\mathbf{u}=\begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \text{ and } \mathbf{v}=\begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} \rightarrow \mathbf{u}^T \mathbf{v} = \mathbf{u}.\mathbf{v} = [u_1 \quad u_2 \quad \cdot \quad \cdot \quad u_n]\begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

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Example 5: Compute u . v and v . u for u=$\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and v=$\begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$?

Solution :

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v} = [\,2 \quad -5 \quad -1\,]\begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} = (2)(3) + (-5)(2) + (-1)(-3) = -1$$

$$\mathbf{v}\cdot\mathbf{u} = \mathbf{v}^T\mathbf{u} = [\,3 \quad 2 \quad -3\,]\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} = (3)(2) + (2)(-5) + (-3)(-1) = -1$$

حساب الزاوية المحصورة مابين متجهين من القانون رقم 2

Example 6 : Find the angle between u = (1, 0, 1) and v = (1, 1, 0)?

Solution : **u · v =** 1*1+0*1+1*0=1

$\|u\|=\sqrt{(1)^2+(0)^2+(1)^2}=\sqrt{2}$, $\|v\|=\sqrt{(1)^2+(1)^2+(0)^2}=\sqrt{2}$ →

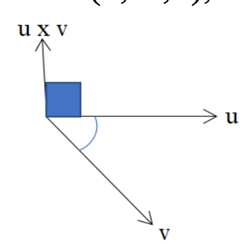
From equation 2 , **cos θ**=$\frac{1}{\sqrt{2}*\sqrt{2}}=\frac{1}{2}$ → θ=$\frac{\pi}{2}$

Cross product

Cross product of two vectors u and v is perpendicular to both the vectors and the plane that contains both the vectors. It is represented by:

$$\mathbf{u \times v = |u|\ |v|\ \sin\theta}$$

Example 7: Find the Cross product for vectors u = (2, -4, 4) and v = (4, 0,3), if the angle between them was 48^o?



Solution :

Cross product = u x v = |u| |v| sin θ

u= 2i - 4j + 4k , v= 4i + 0j +3k

$|u|=\sqrt{(2^2+4^2+4^2)}=\sqrt{36}=6$

$|v|=\sqrt{(4^2+0^2+3^2)}=\sqrt{25}=5$ → **u x v = 6 * 5 *** 0.743≈ 22.29

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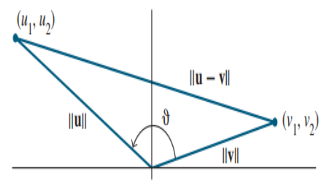
Orthogonal vectors

Two vectors u and v in R^n are orthogonal متعامدان ($\theta=90^o$) if :



$$\mathbf{u} \cdot \mathbf{v} = 0$$

Note: $\mathbf{u.v} = ||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta$

Example 8: Determine which pairs of vectors are orthogonal?

1. u= $\begin{bmatrix} 8 \\ -5 \end{bmatrix}$ and v= $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$ Solution : u.v=(8)(−2)+(−5)(−3)=-16+15=-1 → u.v≠0
   The vectors are NOT orthogonal.

2. u= $\begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$ and v= $\begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$ Solution : u.v=(3)(−4)+(2)(1)+(−5)(−2)+(0)(6)=-12+2+10=0

The vectors are orthogonal

Orthonormal vectors (1. unit vector , 2. orthogonal)شروطها

A set of vectors ($\mathbf{u}_1$, ……, $\mathbf{u}_n$) is **orthonormal set** if it is orthogonal set of unit vectors.

Example 9: Show that (v_1, v_2, v_3) is an orthonormal basis of R^3, where:

$$\mathbf{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

Solution 1) Now compute if there is unit vector or not(from equation 1) :

$$||\mathbf{v}_1|| = \sqrt{(\tfrac{3}{\sqrt{11}})^2 + (\tfrac{1}{\sqrt{11}})^2 + (\tfrac{1}{\sqrt{11}})^2} = \sqrt{\tfrac{9}{11} + \tfrac{1}{11} + \tfrac{1}{11}} = \sqrt{\tfrac{11}{11}} = 1$$

$$||\mathbf{v}_2|| = \sqrt{(\tfrac{-1}{\sqrt{6}})^2 + (\tfrac{2}{\sqrt{6}})^2 + (\tfrac{1}{\sqrt{6}})^2} = \sqrt{\tfrac{1}{6} + \tfrac{4}{6} + \tfrac{1}{6}} = \sqrt{\tfrac{6}{6}} = 1$$

$$||\mathbf{v}_3|| = \sqrt{(\tfrac{-1}{\sqrt{66}})^2 + (\tfrac{-4}{\sqrt{66}})^2 + (\tfrac{7}{\sqrt{66}})^2} = \sqrt{\tfrac{1}{66} + \tfrac{16}{66} + \tfrac{49}{66}} = \sqrt{\tfrac{66}{66}} = 1$$ → $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are unit vectors

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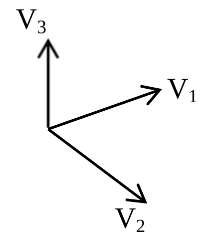

2) Now check if v_1 ,v_2 , v_3 are orthogonal:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = -3/\sqrt{66} + 2/\sqrt{66} + 1/\sqrt{66} = 0$$
$$\mathbf{v}_1 \cdot \mathbf{v}_3 = -3/\sqrt{726} - 4/\sqrt{726} + 7/\sqrt{726} = 0$$
$$\mathbf{v}_2 \cdot \mathbf{v}_3 = 1/\sqrt{396} - 8/\sqrt{396} + 7/\sqrt{396} = 0$$

Thus $\{v_1, v_2, v_3\}$ is **orthogonal** set → $\{\mathbf{v}_1, \mathbf{v}_2,$ and $\mathbf{v}_3\}$ is an **orthonormal** set

HW 1

A. Two vectors **u** = $\begin{bmatrix}1\\-2\end{bmatrix}$ and **v** = $\begin{bmatrix}\mathbf{2}\\\mathbf{-5}\end{bmatrix}$ in R^2, find 4 u , (-3v) , 4u+(-3)v ?

B. Find the angle between u = i + j + k and v = 2i + j – k by using dot product (u.v) ?

C. Determine which the vectors are orthogonal or not?

1. u= $\begin{bmatrix}4\\-2\\5\end{bmatrix}$ and v= $\begin{bmatrix}11\\-1\\-9\end{bmatrix}$ 2. u= $\begin{bmatrix}3\\-6\\7\\8\end{bmatrix}$ and v= $\begin{bmatrix}-9\\6\\17\\-7\end{bmatrix}$

3. u= $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ and v= $\begin{bmatrix}0\\1\\2\end{bmatrix}$ 4. u= $\begin{bmatrix}2\\-7\\-1\end{bmatrix}$, v= $\begin{bmatrix}-6\\-3\\9\end{bmatrix}$, w= $\begin{bmatrix}3\\1\\-1\end{bmatrix}$

5. u= $\begin{bmatrix}2\\-5\\-3\end{bmatrix}$, v= $\begin{bmatrix}0\\0\\0\end{bmatrix}$, w= $\begin{bmatrix}4\\2\\6\end{bmatrix}$

D. Determine which the following vectors are orthonormal or not?

1. u= $\begin{bmatrix}1\\5\\0\end{bmatrix}$, v= $\begin{bmatrix}4\\1\\6\end{bmatrix}$, w= $\begin{bmatrix}1\\0\\1\end{bmatrix}$, 2. u= $\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}$, v= $\begin{bmatrix}\frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}$, w= $\begin{bmatrix}0\\0\\1\end{bmatrix}$

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