Differentiation

Definition of derivative:

The derivative is the rate of change of a function with respect to its variable, or the slope of the tangent line to the curve at a given point.

المشتقة هي معدل تغيّر دالة بالنسبة لمتغيّرها، أو هي ميل المماس للمنحنى عند نقطة معيّنة.

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

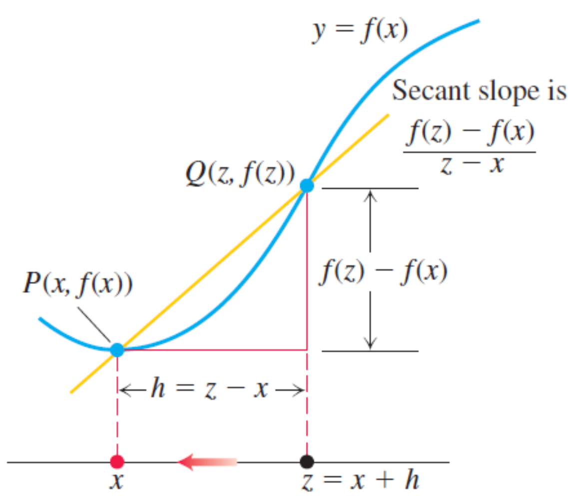
صيغة بديلة:

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

If we write z = x + h, then h = z - x and h approaches 0 if and only if z approaches x.

تقترب h من 0 إذا وفقط إذا اقتربت z من x.



Example 1: Find the slope of the curve y = 1/x at any point x = a ≠ 0. What is the slope at the point x = -1?

Solution:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\frac{1}{a+h}-\frac{1}{a}}{h}=\lim_{h\to 0}\frac{1}{h}\,\frac{a-(a+h)}{a(a+h)}$$

$$=\lim_{h\to 0}\frac{-h}{ha(a+h)}=\lim_{h\to 0}\frac{-1}{a(a+h)}=-\frac{1}{a^2}.$$

Example 2 : Applying the Definition, differentiate $f(x)=\dfrac{x}{x-1}$

Solution We use the definition of derivative, which requires us to calculate $f(x+h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x)=\frac{x}{x-1}\quad\text{and}\quad f(x+h)=\frac{(x+h)}{(x+h)-1},\ \text{so}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x+h}{x+h-1}-\frac{x}{x-1}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\cdot\frac{(x+h)(x-1)-x(x+h-1)}{(x+h-1)(x-1)}$$

$$=\lim_{h\to 0}\frac{1}{h}\cdot\frac{-h}{(x+h-1)(x-1)}$$

$$=\lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)}=\frac{-1}{(x-1)^2}.$$

Example 3:

(a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

(a) We use the alternative formula to calculate f':

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\ &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{\left(\sqrt{z} - \sqrt{x}\right)\left(\sqrt{z} + \sqrt{x}\right)} && \frac{1}{a^2 - b^2} = \frac{1}{(a - b)(a + b)} \\ &= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. && \text{Cancel and evaluate.} \end{aligned}$$

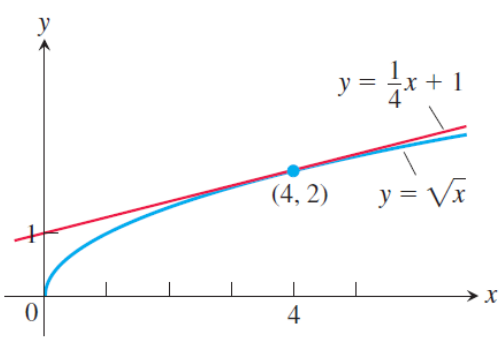
(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point (4, 2) with slope 1/4 (Figure 3.5):

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1.$$



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Differentiation Rules:

If f(x) or y is differentiable function of X and C is constant then:

1- $\frac{d}{dx}\,(c) = 0$

2- $\frac{d}{dx}\,(c\,f(x)) = c\,\frac{d\,f(x)}{dx}$

3- $\frac{d}{dx}\,(f(x) \mp g(x)) = \frac{d\,f(x)}{dx} \mp \frac{d\,g(x)}{dx} = f'(x) \mp g'(x)$

4- $\frac{d}{dx}\,(f(x)\,g(x)) = f(x)\,g'(x) + g(x)f'(x)$

5- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)\,g'(x)}{(g(x))^2}$

6- $\frac{d}{dx}\,(x^n) = n\,x^{n-1}$

7- $\frac{d}{dx}\,(f(x))^n = n\,((f(x))^{n-1}\,.\,f'(x)$

Example 4:

Differentiate the following powers of *x:*

(a) x^3 **(b)** $x^{2/3}$ **(c)** $x^{\sqrt{2}}$ **(d)** $\frac{1}{x^4}$

Solution

(a) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$

(b) $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$

(c) $\frac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1}$

(d) $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$

Example 5: Find the derivative of the functions **:**

1- $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

2- $f(x) = x^7(x^5 + x^2)$

$$f'(x) = x^7(5x^4 + 2x) + 7x^6(x^5 + x^2)$$

3- $f(x) = \frac{5}{x}$, $f(x) = 5x^{-1}$, $f'(x) = -5x^{-2} = -\frac{5}{x^2}$

4- $f(x) = \frac{3x^2 - 5x}{7}$, $f'(x) = \frac{1}{7}(6x - 5)$

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f''= (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative.

إذا كانت الدالة $y = f(x)$ قابلة للاشتقاق، فإن مشتقتها $f'(x)$ هي أيضًا دالة. وإذا كانت f' قابلة للاشتقاق أيضًا، فيمكننا اشتقاق f' للحصول على دالة جديدة لـ x نرمز لها بـ f''. إذن $f'' = (f')$. تُسمى الدالة f'' بالمشتقة الثانية للدالة f لأنها مشتقة المشتقة الأولى.

It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30x^4.$$

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of y with respect to x.

The names continue as you imagine, with $$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

denoting t

he ***n*th derivative** of y with respect to x for any positive integer n.

Example 6:

The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

Derivatives of Trigonometric Functions

Because sin x and cos x are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

1- $\dfrac{d}{dx}(\sin u) = \cos u.\dfrac{du}{dx}$

2- $\dfrac{d}{dx}(\cos u) = -\sin u.\dfrac{du}{dx}$

3- $\dfrac{d}{dx}(\tan u) = \sec^2 u.\dfrac{du}{dx}$

4- $\dfrac{d}{dx}(\cot u) = -\csc^2 u.\dfrac{du}{dx}$

5- $\dfrac{d}{dx}(\sec u) = \sec u \tan u.\dfrac{du}{dx}$

6- $\dfrac{d}{dx}(\csc u) = -\csc u \cot u.\dfrac{du}{dx}$

Example 7 :

a) $y = x^2 \sin x$:

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + 2x \sin x$$
$$= x^2 \cos x + 2x \sin x.$$

b) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$
$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$
$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$
$$= \frac{1}{1 - \sin x}$$

c) Find y'' if $y = \sec x$.

$$y = \sec x$$
$$y' = \sec x \tan x$$
$$y'' = \frac{d}{dx}(\sec x \tan x)$$
$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x)$$
$$= \sec x (\sec^2 x) + \tan x (\sec x \tan x)$$
$$= \sec^3 x + \sec x \tan^2 x$$

d) $y = \sin(x^5 - x)$

$y' = [\cos(x^5 - x)].(5x^4 - 1)$

e) $y = \cos\left(\frac{x}{x+2}\right)$

$y' = -\sin\left(\frac{x}{x+2}\right)\left(\frac{(x+2)-x}{(x+2)^2}\right) = -\left(\frac{2}{(x+2)^2}\right)\sin\left(\frac{x}{x+2}\right)$

f) $y = \sin^5 x = (\sin x)^5$

$y' = 5(\sin x)^4 . \cos x = 5\cos x \sin^4 x$

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EXERCISES:

Q1/ Using the definition, calculate the derivatives of the functions. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2$; $f'(-3)$, $f'(0)$
2. $f(x) = \frac{1-x}{2x}$; $f'(-1)$, $f'(1)$

Q2/ Find the derivatives of the functions

1. $y = (x - 1)(x + 2)$
2. $y = \frac{5x+1}{2\sqrt{x}}$
3. $y = \frac{3}{x} + 5\sin x$
4. $y = \frac{x+3\cot x}{5-2\csc x}$
5. $y = \tan(5 - \sin 2x)$
6. $y = tan^5\, x^4$

Q3/

1. Find y'' if $y = \frac{4x^3}{3} - x$
2. Find $\frac{d^4y}{dx^4}$ if $y = x^6 - 2x^4 + 3\cos x$