



Line Integral in the Complex Plane.

Complex definite integrals are called (complex) **line integrals** written as:

$$\int_C f(z) dz$$

Integrated over a portion of curve C . The function $z(t)=4\cos t + 4i \sin t$ ($-\pi \leq t \leq \pi$) represents the circle.

Basic Properties

$$1. \int_C [k_1 f_1(z) + k_2 f_2(z)] dz = k_1 \int_C f_1(z) dz + k_2 \int_C f_2(z) dz.$$

$$2. \int_{z_0}^Z f(z) dz = - \int_Z^{z_0} f(z) dz.$$

$$3. \int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz.$$



Example 1: find the integrals :

$$\int_0^{1+i} z^2 dz = \frac{z^3}{3} \Big|_0^{1+i} = \frac{(1+i)^3 - (0)^3}{3} = \frac{(1+i)(1+i)^2}{3} = \frac{(1+i)(1+2i+i^2)}{3} = \frac{(1+i)(2i)}{3} = \frac{-2+2i}{3}$$

$$\int_{-\pi i}^{\pi i} \cos z = \sin z \Big|_{-\pi i}^{\pi i} = (\sin \pi i) - (\sin -\pi i) = (\sin \pi i) + (\sin \pi i) = 2 \sin \pi i$$

But $(\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}))$ ← take $x = \pi i$

$$\text{Then : } = 2 \sin \pi i = 2 \frac{1}{2i}(e^{i \pi i} - e^{-i \pi i}) = \frac{1}{i}(e^{-\pi} - e^{\pi}) = -\frac{1}{i}(e^{\pi} - e^{-\pi}) = -2 \frac{\sinh \pi}{i} * \frac{i}{i} =$$

$$2i \sinh \pi = 2i * 11.5487 = 23.0974 i$$

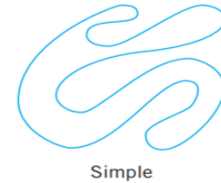
(but $\sinh y = \frac{(e^y - e^{-y})}{2}$, take $y = \pi \rightarrow \sinh \pi = \frac{(e^{\pi} - e^{-\pi})}{2}$)



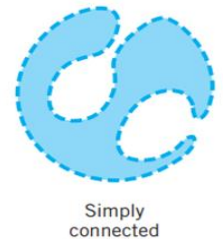
Cauchy's Integral Formula.

Line integral of a function generally depends not only on the endpoints but also on the choice of the path.

❖ A **simple closed path**: No intersect or touch itself



❖ A **simply connected domain** D : Every simple closed path in D encloses only points of D



Theorem 1 (Cauchy's Integral Formula)

If it is analytic in a simply connected domain D , then for every simple closed path:

$$\oint_C f(z) dz = 0.$$

A simple closed path is sometimes called a **contour** and an integral over such a path a **contour integral**.

Example 2 : **Entire Functions**

$$\oint_C e^z dz = 0, \quad \oint_C \cos z dz = 0$$

Points Outside the Contour $\rightarrow f(x)$ Not Analytic

$$\oint_C \sec z dz = 0, \quad \sec z = 1/\cos z \text{ is not analytic at } z = \pm\pi/2, \pm3\pi/2$$

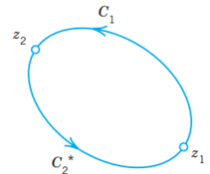
$$\oint_C \frac{dz}{z^2 + 4} = 0 \quad \text{not analytic at } z = \pm 2i, \text{ outside } C$$



Independence of Path

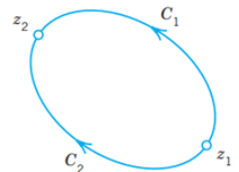
The integral of $f(z)$ is independent of path in D

$$\int_{C_1} f dz + \int_{C_2^*} f dz = 0, \quad \text{thus} \quad \int_{C_1} f dz = - \int_{C_2^*} f dz$$



Sign on the right disappears if we integrate in the reverse direction

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$$



Note : The **unit circle** mean $|z|=1$

دائرة الوحدة هي جميع الأعداد المركبة z التي تحقق $|z|=1$

Example 3 :

Integrate $f(z) = \exp(-z^2)$ counterclockwise around the **unit circle**. Indicate whether satisfy **Cauchy's integral theorem**. Show in details

Solution: $\exp(-z^2)$ is **entire** (analytic everywhere) inside and on the unit circle.

الدالة تامة Entire function يعني شاملة لجميع الاعداد لانه:

- لا تحتوي مقام
- لا يوجد نقاط شاذة (Singularities)
- تحليلية في كل مكان في المستوى المركب.

نظرية كوشي تقول:

إذا كانت الدالة تحليلية تامة وعلى المسار المغلق C فإن:

$$\oint_C e^{-z^2} dz = 0 \quad \text{satisfy Cauchy's Integral Theorem}$$



Example 4 : Find singularities for the function $f(z)=\frac{1}{2z-1}$ and Check Cauchy's Theorem? نبحث عن العدد الذي يجعل المقام = 0

Solution :

$$2z - 1 = 0 \Rightarrow z = \frac{1}{2} \quad \longrightarrow \quad \left| \frac{1}{2} \right| = \frac{1}{2} < 1 \quad \text{inside the unit circle.}$$

→ Not analytic inside C → Cannot apply Cauchy's Theorem لان العدد من ضمن مجال الاعداد

Example 5: Find singularities for the function $f(z)=\frac{1}{z^4-1.1}$ and Check Cauchy's Theorem?

Solution : $z^4 - 1.1 = 0$ ايجاد العدد الذي يجعل المقام = 0

$$Z^4=1.1 \rightarrow |z|=(1.1)^{1/4} \approx 1.024 > 1 \quad \text{outside the unit circle.}$$

→ No singularities شاذة قيم inside the unit circle → analytic function

→ Satisfy Cauchy's Integral Theorem

$$\oint_C \frac{1}{z^4 - 1.1} dz = 0$$

Cauchy's integral formula

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

الصيغة العامة لاييجاد قيمة التكامل وفق صيغة كوشي.....

Example 6 : find integrals by using **Cauchy's integral formula**?

$$1. \oint \frac{e^z}{z-2} dz$$

Solution :

$$\oint_C \frac{e^z}{z - 2} dz = 2\pi i e^z \Big|_{z=2} = 2\pi i e^2 = 46.4268i$$



$$2. \oint \frac{z^3-6}{2z-i} dz, \text{ solution : } \oint \frac{z^3-6}{2z-i} dz = \oint \frac{\frac{1}{2}z^3-3}{z-\frac{1}{2}i} dz = 2\pi i \left[\frac{1}{2}z^3 - 3 \right] \Big|_{z=i/2} \\ = \frac{\pi}{8} - 6\pi i$$

$$3. \oint_{|z|=2} \frac{e^z}{z} dz$$

Solution : $f(z)=e^z$ (analytic everywhere) using Cauchy's Integral Formula, $z_0=0$

$$\oint \frac{e^z}{z} dz = 2\pi i f(0), \quad f(0) = e^z = e^0 = 1 \rightarrow \oint \frac{e^z}{z} dz = 2\pi i$$

Derivatives of Analytic Functions

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz$$

الصيغ العامة لاييجاد التكامل من خلال مشتقة الدالة

نأخذ المشتقة الاولى اذا كان المقام تربيع

$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^3} dz$$

ونأخذ المشتقة الثانية اذا كان المقام تكعيب

Example 7 : Evaluate $\oint \frac{z^4-3z^2+6}{(z+i)^3} dz$ by Cauchy's integral formula?

$$\oint \frac{z^4-3z^2+6}{(z+i)^3} dz = \frac{2\pi i}{2!} f''(z) \Big|_{z=-i}$$

تعوض بالمعادلة $f(z)=z^4-3z^2+6$, $f'(z)=4z^3-6z$ and $f''(z)=12z^2-6$

$$\oint \frac{z^4-3z^2+6}{(z+i)^3} dz = \pi i (z^4-3z^2+6)'' \Big|_{z=-i} = \pi i (12z^2-6) \Big|_{z=-i} = \pi i (12(-i)^2-6) = -18\pi i$$



Example 8 : Evaluate $\oint \frac{e^z}{z^2} dz$ by Cauchy's integral formula?

Solution :

$$\oint \frac{e^z}{z^2} dz = \oint \frac{e^z}{(z-0)^2} dz = 2\pi i f'(0) \quad \text{كتابتها بالصيغة العامة}$$

$$f(z) = e^z, \quad f'(z) = e^z \rightarrow f'(0) = e^0 = 1$$

$$\oint \frac{e^z}{(z-0)^2} dz = 2\pi i f'(0) = 2\pi i (1) = 2\pi i$$

HW :

1. Find the integral by using Cauchy's integral formula:

$$\oint_{|z|=1} \frac{z+1}{z-3} dz$$

2. Evaluate the integrals by using the derivative (Cauchy's integral formula) ?

$$\oint \frac{z^3}{(z-2)^2} dz, \quad \oint \frac{\ln(1+z)}{z^2} dz, \quad \oint \frac{z^3+1}{(z-1)^3} dz$$