## Dot product and Cross product

### Dot product

The equation is : Dot product **= u.v = $u^T$ v=** ||**u**|| ||**v**|| **cos θ …….2**

Where : = scalar value without brackets

u and v are vectors in $R^n$

Dot product is inner product of u and v

**$u^T$** called the transpose of u , **transpose column vector (u)** to **row vector**.

For example :

$$u=\begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \text{ and } v=\begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} \rightarrow u^T v = u.v = [u_1 \quad u_2 \quad \cdot \quad \cdot \quad u_n]\begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = u_1v_1+u_2v_2+\dots+u_nv_n$$

Example 1: Compute u . v and v . u for $u=\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $v=\begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$ ?

Solution :

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v} = [2 \quad -5 \quad -1]\begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} = (2)(3) + (-5)(2) + (-1)(-3) = -1$$

$$\mathbf{v}\cdot\mathbf{u} = \mathbf{v}^T\mathbf{u} = [3 \quad 2 \quad -3]\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} = (3)(2) + (2)(-5) + (-3)(-1) = -1$$

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Example 2 : Find the angle between u = (1, 0, 1) and v = (1, 1, 0)?

Solution : **u · v =** 1*1+0*1+1*0=1

$||u||=\sqrt{(1)^2+(0)^2+(1)^2}=\sqrt{2}$ , $||v||=\sqrt{(1)^2+(1)^2+(0)^2}=\sqrt{2} \rightarrow$

From equation 2 , **cos θ**$=\frac{1}{\sqrt{2}*\sqrt{2}}=\frac{1}{2} \rightarrow \theta=\frac{\pi}{3}$



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Example 3: Find the angle between u = i + j + k and v = 2i + j – k?

Solution : u=(1,1,1) , v=(2,1,-1)

**u.v = ||u|| ||v|| cos θ**

**u · v** = 1*2+1*1+1*-1=2 , $||u||=\sqrt{(1)^2+(1)^2+(1)^2}=\sqrt{3}$ ,
$||v||=\sqrt{(2)^2+(1)^2+(-1)^2}=\sqrt{6}$

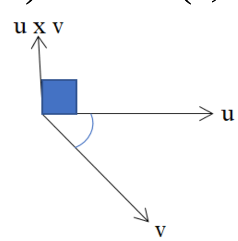
So from equation 1 , **cos θ**$=\frac{2}{\sqrt{3}*\sqrt{6}}=\frac{2}{\sqrt{18}}$ → θ≈61.9°

## Cross product

Cross product of two vectors u and v is perpendicular to both the vectors and the plane that contains both the vectors. It is represented by:

**u x v = |u| |v| sin θ**

Example 4: Find the Cross product for vectors u = (2, -4, 4) and v = (4, 0,3), if the angle between them was $48^o$?




Solution :

Cross product = u x v = |u| |v| sin θ

u= 2i - 4j + 4k , v= 4i + 0j +3k

$|u|=\sqrt{(2^2+4^2+4^2)}=\sqrt{36}=6$

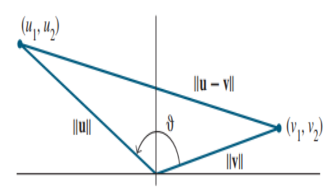
$|v|=\sqrt{(4^2+0^2+3^2)}=\sqrt{25}=5$

→ **u x v = 6 * 5 *** 0.743≈ 22.29

## Orthogonal vectors

Two vectors u and v in $R^n$ are orthogonal متعامدان (**θ**=$90^o$) if :

**u . v = 0**




**Note: u.v = ||u|| ||v|| cos θ**



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Example 5: Determine which pairs of vectors are orthogonal?

1. u=$\begin{bmatrix} 8 \\ -5 \end{bmatrix}$ and v= $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$ Solution : u.v=(8)(−2)+(−5)(−3)=-16+15=-1 $\rightarrow$ u.v≠0
   The vectors are NOT orthogonal.

2. u=$\begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$ and v= $\begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$ Solution : u.v=(3)(−4)+(2)(1)+(−5)(−2)+(0)(6)=-12+2+10=0

The vectors are orthogonal

Example 6: Determine whether the following vectors are orthogonal or not?

u=$\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$ , v= $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$ , w= $\begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$

Solution :

u=(-1,4,-3) , v=(5,2,1) , w=(3,-4,-7)

u.v=(-1)(5)+(4)(2)+(-3)(1)=-5+8-3=0

u.w=(-1)(3)+(4)(-4)+(-3)(-7)=-3-16+21=2

v.w=(5)(3)+(2)(-4)+(1)(-7)=15-8-7=0 →Since u·w≠0 , the vectors are not all orthogonal

## Orthonormal vectors (1. unit vector , 2. orthogonal )شروطها

A set of vectors ($\mathbf{u}_1$, ……, $\mathbf{u}_n$) is **orthonormal set** if it is orthogonal set of unit vectors.

Example 7: Show that ($v_1$, $v_2$, $v_3$) is an orthonormal basis of $R^3$, where:

$$\mathbf{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

Solution 1) Now compute if there is unit vector or not(from equation 1) :



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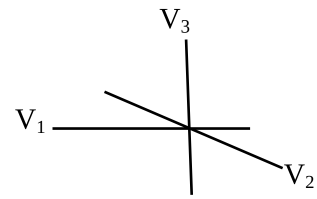
$\|\mathbf{v}_1\|=\sqrt{(\frac{3}{\sqrt{11}})^2+(\frac{1}{\sqrt{11}})^2+(\frac{1}{\sqrt{11}})^2}=\sqrt{\frac{9}{11}+\frac{1}{11}+\frac{1}{11}}=\sqrt{\frac{11}{11}}=\mathbf{1}$

$\|\mathbf{v}_2\|=\sqrt{(\frac{-1}{\sqrt{6}})^2+(\frac{2}{\sqrt{6}})^2+(\frac{1}{\sqrt{6}})^2}=\sqrt{\frac{1}{6}+\frac{4}{6}+\frac{1}{6}}=\sqrt{\frac{6}{6}}=\mathbf{1}$

$\|v_3\|=\sqrt{(\frac{-1}{\sqrt{66}})^2+(\frac{-4}{\sqrt{66}})^2+(\frac{7}{\sqrt{66}})^2}=\sqrt{\frac{1}{66}+\frac{16}{66}+\frac{49}{66}}=\sqrt{\frac{66}{66}}=\mathbf{1}$ $\rightarrow$ $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are unit vectors

2) Now check if $v_1$ ,$v_2$ , $v_3$ are orthogonal:




$$\mathbf{v}_1\cdot\mathbf{v}_2=-3/\sqrt{66}+2/\sqrt{66}+1/\sqrt{66}=0$$
$$\mathbf{v}_1\cdot\mathbf{v}_3=-3/\sqrt{726}-4/\sqrt{726}+7/\sqrt{726}=0$$
$$\mathbf{v}_2\cdot\mathbf{v}_3=1/\sqrt{396}-8/\sqrt{396}+7/\sqrt{396}=0$$

Thus $\{v_1, v_2, v_3\}$ is **orthogonal** set $\rightarrow$ $\{\mathbf{v}_1, \mathbf{v}_2, \text{and } \mathbf{v}_3\}$ is an **orthonormal** set

Example 8 Determine whether the vectors are orthonormal or not? $u=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}$ , $v=\begin{bmatrix}\frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}$ , $w=\begin{bmatrix}0\\0\\1\end{bmatrix}$

Solution : 1) check unit vector

$\|u\|=\sqrt{(\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2+(0)^2}=\sqrt{\frac{1+1}{2}}=1$ , $\|v\|=\sqrt{(\frac{-1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2+(0)^2}=1$

$\|w\|=\sqrt{(0)^2+(0)^2+(1)^2}=1$ $\rightarrow$ the vectors are unit vectors

2) Check Orthogonality

$u\cdot v=\frac{1}{\sqrt{2}}*-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}*\frac{1}{\sqrt{2}}+0*0=0$ , $v\cdot w=\frac{-1}{\sqrt{2}}*0+\frac{1}{\sqrt{2}}*0+0*1=0$

$u.w=\frac{1}{\sqrt{2}}*0+\frac{1}{\sqrt{2}}*0+0*1=0$ $\rightarrow$ the vectors are orthonormal



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**HW 1**

**A.** Two vectors **u** = $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and **v** = $\begin{bmatrix} \mathbf{2} \\ \mathbf{-5} \end{bmatrix}$ in $R^2$, find 4 u , (-3v) , 4u+(-3)v ?

**B.** Determine which the vectors are orthogonal or not?

1. u= $\begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$ and v= $\begin{bmatrix} 11 \\ -1 \\ -9 \end{bmatrix}$

2. u= $\begin{bmatrix} 3 \\ -6 \\ 7 \\ 8 \end{bmatrix}$ and v= $\begin{bmatrix} -9 \\ 6 \\ 17 \\ -7 \end{bmatrix}$

3. u= $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and v= $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

4. u= $\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}$ , v= $\begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}$ , w= $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

5. u= $\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$ , v= $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ , w= $\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$

**C.** Determine the vectors are orthonormal or not?

u= $\begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$ , v= $\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$ , w= $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$



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