Baye's Rule

Suppose $A_1, A_2, \ldots, A_n$ are n mutually exclusive and exhaustive events such that $P(A_i) \neq 0$. Then for $i = 1, 2, 3, \ldots, n$,

$$P\left(\frac{A_i}{A}\right) = \frac{P(A_i)\cdot P\left(\frac{A}{A_i}\right)}{\sum_{k=1}^{n} P(A_K)P\left(\frac{A}{A_K}\right)}$$

Where A is an arbitrary event of S.

شرح اضافي حول Bays theorem

نظرية بايز لها علاقه مباشره مع الاحتماليه المشروطه conditional probability . حيث اذ كانت A1, A2, A3 , ………………..An حوادث شامله وان B هو حدث معين واحتمال وقوعه لا يساوي صفر فأن

$$P(A1\backslash B) = \frac{P(A1).P(B\backslash A1)}{P(A1).P(B\backslash A1) + P(A2).P(B\backslash A2) + \cdots P(An).P(B\backslash An)}$$

مثال:

مصنع يتكون من ثلاثة اقسام لانتاج الحاسبات وكما يلي:

- **ينتج القسم A1 50% من الانتاج الكلي ومتوسط المعاب 3%**
- **ينتج القسم A2 30% من الانتاج الكلي ومتوسط المعاب 4%**
- **ينتج القسم A3 20% من الانتاج الكلي والمعاب 5%**
- **فاذا اخذنا حاسبه واحده بصوره عشوائيه من انتاج المصنع الكلي ووجدت معابه فما هو احتمال ان تكون من انتاج القسم A1**

A factory consists of three departments for the production of computers, as follows:

- **Department A1 produces 50% of the total production, with an average defect rate of 3%**
- **Department A2 produces 30% of the total production, with an average defect rate of 4%**
- **Department A3 produces 20% of the total production, with an average defect rate of 5%**

If we randomly select one computer from the factory's total production and find it - defective, what is the probability that it was produced by Department A1?

Solution:

$$P(A1\backslash D) = \frac{P(A1).P(D\backslash A1)}{P(A1).P(D\backslash A1) + P(A2).P(D\backslash A2) + P(A3).P(D\backslash A3)}$$

Let:

- P(A1)=0.5P(A1) = 0.5 → Probability a computer is from A1
- P(A2)=0.3P(A2) = 0.3
- P(A3)=0.2P(A3) = 0.2

Defect rates:

- P(D|A1)=0.03P(D|A1) = 0.03
- P(D|A2)=0.04P(D|A2) = 0.04

We want:

- $P(A1|D)$ → Probability a defective computer came from A1

Using the law of total probability:

$$P(D) = P(D|A1)P(A1) + P(D|A2)P(A2) + P(D|A3)P(A3)$$

$$P(D) = (0.03)(0.5) + (0.04)(0.3) + (0.05)(0.2) = 0.015 + 0.012 + 0.01 = 0.037$$

Apply Bayes' Theorem

$$P(A1|D) = \frac{P(D|A1)P(A1)}{P(D)} = \frac{(0.03)(0.5)}{0.037} = \frac{0.015}{0.037} \approx 0.405$$

The probability that a randomly selected defective computer was produced b Department A1 is approximately 0.405 or 40.5%.

Want to try this for A2 or A3 as well?

```
Start
├── A1 (50%)
│   ├── Defective (3%) → P(A1 ∩ D) = 0.5 × 0.03 = 0.015
│   └── Non-defective (97%)
├── A2 (30%)
│   ├── Defective (4%) → P(A2 ∩ D) = 0.3 × 0.04 = 0.012
│   └── Non-defective (96%)
└── A3 (20%)
    ├── Defective (5%) → P(A3 ∩ D) = 0.2 × 0.05 = 0.010
    └── Non-defective (95%)
```

MORE EXPLANATION AND SOLVED EXAMPLES ABOUT BAYES THEOREM:

Bayes' Theorem (also known as Bayes' rule) is a deceptively simple formula used to calculate conditional probability. The Theorem was named after English mathematician Thomas Bayes (1701-1761). The formal definition for the rule is:

مبرهنة بايز (المعروفة أيضًا باسم قاعدة بايز) هي صيغة بسيطة للغاية تُستخدم لحساب الاحتمال الشرطي. سُميت هذه المبرهنة تيمنًا بعالم الرياضيات الإنجليزي توماس بايز (1701-1761). التعريف الرسمي للقاعدة هو:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

In most cases, you can't just plug numbers into an equation; You have to figure out what your "tests" and "events" are first. For two events, A and B, Bayes' theorem allows you to figure out p(A|B) (the probability that event A happened, given that test B was positive) from p(B|A) (the probability that test B happened, given that event A happened). It can be a little tricky to wrap your head around as technically you're working backwards; you may have to switch your tests and events around, which can get confusing. An example should clarify what I mean by "switch the tests and events around."

في معظم الحالات، لا يمكنك ببساطة إدخال الأرقام في المعادلة؛ بل عليك أولاً تحديد "الاختبارات" و"الأحداث". بالنسبة للحدثين A وB، تسمح لك نظرية بايز بحساب p(A|B) احتمال وقوع الحدث A، بشرط أن يكون الاختبار B موجبًا) من p(B|A) (احتمال وقوع الاختبار B، بشرط وقوع الحدث A). قد يكون من الصعب فهم الأمر، لأنك من الناحية الفنية تعمل بطريقة عكسية؛ فقد تضطر إلى تبديل الاختبارات والأحداث، مما قد يُسبب لك بعض الالتباس. سيوضح مثال ما أقصده بـ "تبديل الاختبارات والأحداث".

You might be interested in finding out a patient's probability of having liver disease if they are an alcoholic. "Being an alcoholic" is the **test** (kind of like a litmus test) for liver disease.

- **A** could mean the event "Patient has liver disease." Past data tells you that 10% of patients entering your clinic have liver disease. P(A) = 0.10.
- **B** could mean the litmus test that "Patient is an alcoholic." Five percent of the clinic's patients are alcoholics. P(B) = 0.05.
- You might also know that among those patients diagnosed with liver disease, 7% are alcoholics. This is your **B|A:** the probability that a patient is alcoholic, given that they have liver disease, is 7%.
  Bayes' theorem tells you:

  P(A|B) = (0.07 * 0.1)/0.05 = 0.14

  In other words, if the patient is an alcoholic, their chances of having liver disease is 0.14 (14%). This is a large increase from the 10% suggested by past data. But it's still unlikely that any particular patient has liver disease.

بمعنى آخر، إذا كان المريض مدمنًا على الكحول، فإن احتمال إصابته بأمراض الكبد هو 0.14 (14%). وهذه زيادة كبيرة عن نسبة الـ 10% التي أشارت إليها البيانات السابقة. ولكن لا يزال من غير المرجح أن يُصاب أي مريض بأمراض الكبد.

Example 10

Akshay speaks the truth in 45% of the cases, In a rainy season, on each day there is a 75% chance of raining. On a certain day in the rainy season, Akshay tells his mother that it is raining outside. What is the probability that it is actually raining?

Solution

Let E denote the event that it is raining and A denote the event that Akshay tells his mother that it is raining outside.

Then, $P(E) = \frac{3}{4}$, $P(\bar{E}) = \frac{1}{4}$

$$P\left(\frac{A}{E}\right) = \frac{45}{100} = \frac{9}{20} \text{ and } P\left(\frac{A}{\bar{E}}\right) = \frac{11}{20}$$

By Baye's Rule, we have

$$P\left(\frac{E}{A}\right) = \frac{P(E)P\left(\frac{A}{E}\right)}{P(E)P\left(\frac{A}{E}\right) + P(\bar{E})P\left(\frac{A}{\bar{E}}\right)}$$

$$= \frac{\frac{3}{4} \times \frac{9}{20}}{\frac{3}{4} \times \frac{9}{20} + \frac{1}{4} \times \frac{11}{20}} = \frac{27}{38}.$$

c c c c

Example 11:

A person has undertaken a job. The probability of completing the job on time if it rains is 0.44, and the probability of completing the job on time if it does not rain is 0.95. If the probability that it will rain is 0.45, then determine the probability that the job will be completed on time.

Solution:

Let:

- *R: event that it rains*
- R^c*: event that it does not rain*
- *C: event that the job is completed on time*

We are given:

$$P(R) = 0.45, P(Rc) = 1 - 0.45 = 0.55$$

$$P(C \mid R) = 0.44, P(C \mid Rc) = 0.95$$

By the law of total probability:

$$P(C) = P(R)P(C \mid R) + P(Rc)P(C \mid Rc)$$

Substitute values:

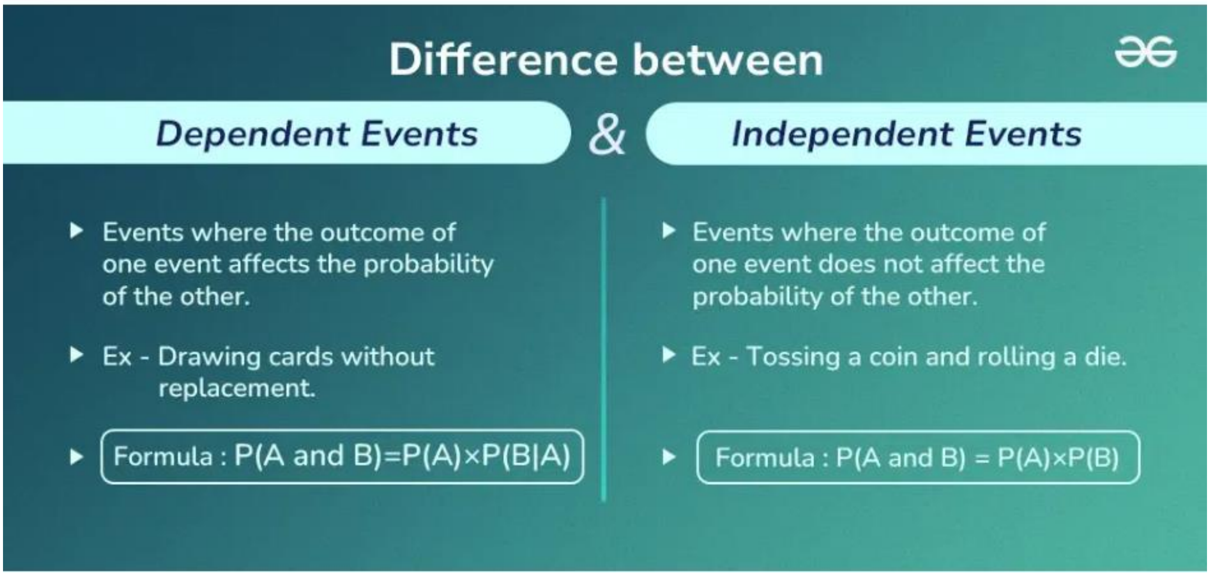
$$P(C) = (0.45)(0.44) + (0.55)(0.95)$$

$$P(C) = 0.198 + 0.5225 = 0.7205$$

Dependent and Independent Events in Probability

An event in probability falls under two categories,

- Dependent Events
- Independent Events



Examples of Dependent Events

For Example, let's say three cards are to be drawn from a pack of cards. Then the probability of getting a king is highest when the first card is drawn, while the probability of getting a king would be less when the second card is drawn. In the draw of the third card, this probability would be dependent upon the outcomes of the previous two cards. We can say that after drawing one card, there will be fewer cards available in the deck, therefore the probabilities after each drawn card changes.

Independent Events

Independent events are those events whose occurrence is not dependent on any other event. If the probability of occurrence of an event A is not affected by the occurrence of another event B, then A and B are said to be independent events.

Examples of Independent Events

Various examples of Independent events are:

- **Tossing a Coin**

Sample Space(S) in a Coin Toss = {H, T}
Both getting H and T are Independent Events

- **Rolling a Die**

Sample Space(S) in Rooling a Die = {1, 2, 3, 4, 5, 6}, all of these events are independent too.

Both of the above examples are simple events. Even compound events can be independent events. For example:

- **Tossing a Coin and Rolling a Die**

If we simultaneously toss a coin and roll a die then the probability of all the events is the same and all of the events are independent events,

Sample Space(S) of such experiment = {(1, H), (2, H), (3, H), (4, H), (5, H), (6, H), (1, T), (2, T), (3, T), (4, T) (5, T) (6, T)}.
These events are independent because only one can occur at a time and occurring of one event does not affect other events.

Note:

- A and B are two events associated with the same random experiment, then A and B are known as independent events if

$$P(A \cap B) = P(B).P(A)$$

Mutually Exclusive Events

Two events A and B are said to be mutually exclusive events if they cannot occur at the same time. Mutually exclusive events never have an outcome in common. If we take two events A and B as mutually exclusive events where the probability of event A is P(A) and the probability of event B is P(B) then the probability of happening both events together is,

$P(A \cap B) = 0$

Then the probability of occurring any one event is,

$$P(AUB) = P(A) \text{ or } P(B) = P(A) + P(B)$$

Examples of Mutually Exclusive Events

Some examples of mutually exclusive events are,

- Tossing a coin we either get a head or a tail. Head and tail cannot appear simultaneously. Therefore, the occurrence of a head or a tail is two mutually exclusive events.
- In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive as if any one of these faces comes on the top, the possibility of others in the same trial is ruled out.

bbb

Solved Examples on Dependent and Independent Events:

Example 1: An instructor has a question bank with 300 Easy T/F, 200 Difficult T/F, 500 Easy MCQ, and 400 Difficult MCQ. If a question is selected randomly from the question bank, What is the probability that it is an easy question given that it is an MCQ?

Solution:

Total question in the question bank = 300 + 200 + 500 + 400

P(Easy) = (300+500)/1400 = 800/1400 = 4/7

P(MCQ) = (400+500)/1400 = 900/1400 = 9/14

P(Easy ∩ MCQ) = (500)/1400 =5/14

P(Easy|MCQ) = P(Easy ∩ MCQ)/P(MCQ)
P(Easy|MCQ) = (5/14)/(9/14) = 5/9
Thus, probability of an easy question given it is an MCQ is 5/9.

Example 2: In a shipment of 20 apples, 3 are rotten. 3 apples are randomly selected. What is the probability that all three are rotten if the first and second are not replaced?
Solution:
Total Apple = 20
Rotten Apple = 3

- *Possibility of the first apple being rotten = 3/20*
- *Possibility of the second apple being rotten = 2/19*
- *Possibility of the third apple being rotten = 1/18*

Probability of all three apples being rotten = P(3 Rotten) = (3/20 × 2/19 × 1/18) = 6/6840 = 1/1140
Thus, probability that all three apples are rotten is, 1/1140

Example 3: John has to select two students from a class of 10 girls and 15 boys. What is the probability that both students chosen are boys?
Solution:
Total number of students = 10 + 15 =25
Probability of choosing the first boy
P (Boy 1) = 15/25
P (Boy 2) = 14/24
P (Boy 1 and Boy 2) = P (Boy 1) and P (Boy 2)
P (Boy 1 and Boy 2) = (15/25) × (14/24) = 7/20
Thus, probability of choosing both boys is 7/20

Example 4: A multiple-choice test consists of two problems. Problem 1 has 5 options and Problem 2 has 4 options. Each problem has only one correct answer. What is the probability of randomly guessing the correct answer to both problems?
Solution:
Here, the probability of the correct answer to Problem 1 = P(A) and the probability of the correct answer to Problem 2 = P(B) are independent events. Thus the probability of a correct answer to Problem 1 and Problem 2 both = P(A ∩ B) = P(A).P(B)

- *P(A) = 1/5*
- *P(B) = 1/4*

P(A ∩ B) = (1/5) × (1/4) = 1/20
Thus, probability of getting both answers correct is 1/20.

الاحتمال الشرطي:

اذا كان E2 , E1 حدثان في فضاء العينه فان احتمال وقوع الحدث E1 مسبوقا بوقوع الحدث E2 (اي ان احتمال وقوع الحدث E2 مشروط بوقوع الحدث E1 وبالعكس وحسب القانون الرياضي التالي:

$$P(E1\backslash E2) = \frac{P(E1.E2)}{P(E2)} = \frac{P(E1).P(E2)}{P(E2)}$$

$$P(E2\backslash E1) = \frac{P(E1.E2)}{P(E1)} = \frac{P(E1).P(E2)}{P(E1)}$$

Example:

لو فرضنا لدينا مخزن فيه عدد 10 حاسبات لاغراض حسابيه و 8 حاسبات لاغراض رسم فأذا تم سحب حاسبتان على التوالي وبدون ارجاع . ماهو احتمال ان تكون الحاسبه الثانيه حاسبة حساب مع العلم ان الحاسبه الاولى ايضا كانت حاسبة حساب.

الحل:

نفرض ان الحدث A حاسبة حساب و الحدث B حاسبة رسم اذن:

Suppose we have a warehouse containing 10 computers for calculations and 8 computers for drawing. If two computers are withdrawn consecutively and are not returned, what is the probability that the second computers is for calculations, given that the first computers was also a calculations?

Let's assume that event A is a calculation computer and event B is a drawing computer. Then:

In case of conditional probability then:

$$P(B\backslash A) = \frac{P(A.B)}{P(A)} = \frac{P(A).P(B)}{P(A)}$$

After the first **calculation computer** is withdrawn:

- Remaining calculation computers: 10−1=910 - 1 = 9
- Remaining total computers: 18−1=17

So the probability that the second computer is also for calculations is:

P=9/17

If we want to explore what happens if the first computer was for drawing instead?

After removing one **drawing computer**:

- Remaining calculation computers: still 10
- Remaining total computers: $18 - 1 = 17$

So the probability that the second computer is for calculations is:

$$P = \frac{10}{17}$$

ADDITIONAL EXAMPLES:

EXAMPLE

A fair die is thrown. List the sample space of the experiment and hence find the probability of observing: (a) a multiple of 3 (b) an odd number.

Are these events mutually exclusive?

Solution

(a) The sample space is $U = \{1, 2, 3, 4, 5, 6\}$.

Let A be the event 'obtaining a multiple of 3'.

We then have that $A = \{3, 6\}$. Therefore, $P(A) = \frac{n(A)}{n(U)} = \frac{2}{6} = \frac{1}{3}$.

(b) Let B be the event 'obtaining an odd number'.

Here $B = \{1, 3, 5\}$ and so $P(B) = \frac{n(B)}{n(U)} = \frac{3}{6} = \frac{1}{2}$.

In this case, $A = \{3, 6\}$ and $B = \{1, 3, 5\}$, so that $A \cap B = \{3\}$. Therefore, as $A \cap B \neq \varnothing$ A and B are not mutually exclusive.

Example :

1	2	3	4	5	6	7	8	9	10

A ⟶ Even B ⟶ Greater than 5

P (A ∪ B) = P (A happening) + P (B happening) − P (A & B happening together)

$= \frac{5}{10} + \frac{5}{10} - \frac{3}{10}$ A --> 2 , 4 , 6 , 8 , 10 B --> 6 , 7 , 8 , 9 , 10

= 0.7

2 , 4 , 6 , 7 , 8 , 9 , 10

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

1	2	3	4	5	6

A ⟶ Odd

B ⟶ Even

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A & B ---> Mutually Exclusive

$P(A \cup B) = P(A) + P(B)$

$= \frac{3}{6} + \frac{3}{6} = 1$

$P(A \cup B) = P(A) + P(B)$ Mutually Exclusive

EXAMPLE

A bag has 20 coins numbered from 1 to 20. A coin is drawn at random and its number is noted. What is the probability that the coin has a number that is divisible by 3 or by 5?

Solution

Let T denote the event "The number is divisible by 3" and S, the event "The number is divisible by 5".

Using the addition rule we have $P(T \cup S) = P(T) + P(S) - P(T \cap S)$

Now, $T = \{3, 6, 9, 12, 15, 18\}$ and $S = \{5, 10, 15, 20\}$ so that $T \cap S = \{15\}$.

Therefore, we have $P(T) = \frac{6}{20}$ and $P(S) = \frac{4}{20}$ and $P(T \cap S) = \frac{1}{20}$.

This means that $P(T \cup S) = \frac{6}{20} + \frac{4}{20} - \frac{1}{20} = \frac{9}{20}$.

EXAMPLE

If $p(A) = 0.6$, $p(B) = 0.3$ and $p(A \cap B) = 0.2$, find

(a) $p(A \cup B)$ (b) $p(B')$

Solution

(a) Using the addition formula we have, $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

$\Rightarrow p(A \cup B) = 0.6 + 0.3 - 0.2 = 0.7$

(b) Using the complementary formula, we have $p(B') = 1 - p(B) = 1 - 0.3 = 0.7$.

Example: bag contain green balls and yellow balls. if you are going to choose two balls with out replacement. if the probability of selecting green ball and yellow ball is 14/39. what is the probability of selecting yellow in the second draw if we know the probability of selecting green ball on the first draw is4/9?

Solution: