Matrix

Linear equation is an equation that can be written in the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Where : b , a_1, …., a_n = real or complex numbers, for example:

$$3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$ المعادلة الخطية

While : $4x_1 - 5x_2 = x_1x_2$ and $x_2 = 2\sqrt{x_1} - 6$ المعادلة الغير خطية

None linear because (x_1x_2, $\sqrt{x_1}$)

System of linear equations (or a **linear system**):

A collection مجموعة of one or more linear equations involving same variables, for example:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned}$$

Matrix : The rectangular array contain information of a linear system.

يحتوي الترتيب المتعامد(المصفوفة) على معلومات النظام الخطي

For previous example , the coefficients معاملات of each variable aligned in columns, as below:

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

Called the **coefficient matrix** (or **matrix of coefficients**) مصفوفة المعاملات

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Called the **augmented matrix** المصفوفة الموسعة

Square Matrix:

Number of rows صفوف= Number of columns الاعمدة. **Example :** $X = \begin{bmatrix} 2 & -7 & 7 \\ 2 & 5 & 6 \\ 5 & 4 & 3 \end{bmatrix}$

Diagonal Matrix

A **diagonal matrix** is a square matrix have:

Elements a_{ij} **≠ 0** when i=j such as (a_{11}, a_{22}, a_{33}) , (diagonal elements)عناصر قطر المصفوفة .

Diagonal matrix of order "3 × 3":

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \text{ for example } \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elements $\mathbf{a_{ij} = 0}$ **when i ≠ j** , (a_{12} a_{21} , a_{13} , a_{31}). named (non-diagonal entries المدخلات)

Zero matrix: m x n matrix, all entries كل المدخلات(diagonal and non-diagonal) =0.

Other Example of a Diagonal Matrix

Diagonal Matrix of order (4 × 4)

$$D_{4,4} = \begin{bmatrix} 22 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & -34 \end{bmatrix}$$

Sum and product of two diagonal matrices also diagonal.

جمع وضرب مصفوفتان قطرية =مصفوفة قطرية

$$\text{If } A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 8 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{, then } AB = \begin{bmatrix} -24 & 0 \\ 0 & -15 \end{bmatrix}$$

- The determinant value can be determined only for square matrices.
- **Rectangular matrix:** m equations(rows)≠ n unknowns المجاهيل (columns).

Examples of diagonal matrices:

- **Scalar عدد ثابت matrices**: $a_{ij}=0$ for $i \neq j$ and $a_{ij}=k$ for $i=j$, when $a_{ij}=k=$ scalar value.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$ The scalar k=5

- **Identity matrix**(I or I_n): square matrix has **(1)** on main diagonal and 0s elsewhere.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **A null matrix** (or **zero matrix**), denoted by 0 with any size (m x n) where every entry is zero.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- A **symmetric** matrix is a matrix A such that **transpose A^T = A**. (necessary square). **Equal** numbers on opposite sides of the main diagonal. تساوي الارقام لجانبي القطر

Example 1 : which of the following matrices are symmetric and non-symmetric?

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & -7 \end{bmatrix}$$ Solution : **Symmetric**

$$\begin{bmatrix} 1 & -3 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -4 & 0 \\ -6 & 1 & -4 \\ 0 & -6 & 1 \end{bmatrix}$$ Solution **: Non-symmetric**

Transpose of a Matrix منقول المصفوفة

Given **m x n** matrix A, transpose of A is the n x m matrix, denoted by A^T , columns are formed from the corresponding rows of A. الصف يصبح عمود والعمود يصبح صف

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

Example 2 : Find the transpose of the following matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}$$

Solution :

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$$

Triangular

A triangular matrix is a square matrix (n x n) where all entries **above or below** the main diagonal are zero.

Upper triangular matrices have zeros below the diagonal. $\begin{pmatrix} 1 & 5 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{pmatrix}$

Lower triangular matrices have zeros above the diagonal. $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 2 & 6 & 3 \end{pmatrix}$

Matrix operation

Addition الجمع

If A and B are m x n matrices, then the **sum** A + B is the m x n matrix whose columns are the sums of the corresponding columns in A and B. The sum A + B is **defined** only when A and B are the **same size**. يجب ان تكون المصفوفتان متساويتان بعدد الصفوف والاعمدة

Theorem: Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$
b. $(A + B) + C = A + (B + C)$
c. $A + 0 = A$
d. $r(A + B) = rA + rB$
e. $(r + s)A = rA + sA$
f. $r(sA) = (rs)A$

Example 3 : Find A + B , and A + C if:

$$A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

Solution :

$$A + B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$$

but A + C is not defined غير معرفة because A and C different sizes

Scalar multiplication

If r is a scalar and A is a matrix, then the **scalar multiple** rA.

Example 4 : If A $=\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$ and B$=\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$ are the matrices , find 2B , A-2B

Solution :

$$2B = 2\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$$

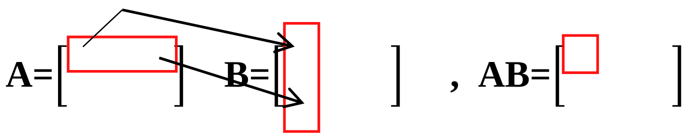
$$A - 2B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}$$

Multiplication الضرب

Row–Column Rule for Computing *AB*

صف عمود

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$



ضرب الارقام داخل المستطيل الافقي للمصفوفة الاولى بالأرقام داخل المستطيل العمودي في المصفوفة الثانية فيعطي الرقم الموجود داخل المربع الاحمر في المصفوفة الناتجة(المصفوفة الثالثة)

Example 5: if A=$\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$ B=$\begin{bmatrix} 2 & 4 \\ 7 & 5 \end{bmatrix}$ find AB?

Solution : A=$\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$ B=$\begin{bmatrix} 2 & 4 \\ 7 & 5 \end{bmatrix}$, AB=$\begin{bmatrix} C & D \\ E & F \end{bmatrix}$

C=1*2+2*7=16 , D=1*4+2*5=14 , E=0*2+6*7=42 , F=0*4+6*5=30

→ AB=$\begin{bmatrix} 16 & 14 \\ 42 & 30 \end{bmatrix}$

Example 6: Compute AB, where:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

Solution :

A=$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ B=$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$, AB=$\begin{bmatrix} C & D & G \\ E & F & H \end{bmatrix}$

C=2*4+3*1= 11 , D=2*3+3*(-2)= 0 , G=2*6+3*3= 21 , E=1*4+(-5) *1=-1

F=1*3+(-5)*(-2)=13 , H=1*6+(-5)*3=9 → **AB=$\begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & 9 \end{bmatrix}$**

Example 7: **Find AB, where:** $A = \begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$

Solution :

هنالك اربع صفوف في المصفوفة الاولى يعني اربع صفوف في المصفوفة الناتجة , وهنالك عمودان (اللذان سيتم الضرب بهما) في المصفوفة الثانية يعني سيكون هنالك عمودان في المصفوفة الناتجة.

يعني تختار عدد الصفوف من المصفوفة الاولى وعدد الاعمدة من المصفوفة الثانية.

AB= $\begin{bmatrix} c & g \\ d & h \\ e & i \\ f & j \end{bmatrix}$,

c=2(4)+(-5)(7)+0(3)=-27 , g=2(-6)+(-5)(1)+0(2)=-17 , d=(-1)(4)+3(7)+(-4)(3)=5

h=(-1)(-6)+3(1)+(-4)(2)=1 , e=6(4)+(-8)(7)+(-7)(3)=-53 , i=6(-6)+(-8)(1)+(-7)(2)=-58

f=(-3)(4)+0(7)+9(3)=15 , j=(-3)(-6)+0(1)+9(2)=36 $\rightarrow$ **AB=** $\begin{bmatrix} -27 & -17 \\ 5 & 1 \\ -53 & -58 \\ 15 & 36 \end{bmatrix}$

Theorem

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

HW / find A*B

Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$

Determinant of Matrices محدد المصفوفات

Determinant of the matrix can be calculate by the following manner:

For matrix 2 x 2 : A= $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

المحدد =حاصل ضرب الطرفين – حاصل ضرب الوسطين

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

Example 8: find the determinant of the matrix **A=** $\begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{-5} \end{bmatrix}$?

Solution : det A=2*(-5)-3*1=-13

For matrix 3 x 3 : A= $\begin{bmatrix} +a_{11} & -a_{12} & +a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

det A= $\mathbf{a_{11}}$ **det** $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ **-** $\mathbf{a_{12}}$ **det** $\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$ **+** $\mathbf{a_{13}}$ **det** $\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

$$= a_{11}(a_{22}*a_{33}-a_{23}*a_{32})-a_{12}(a_{21}*a_{33}-a_{23}*a_{31})+a_{13}(a_{21}*a_{32}-a_{22}*a_{31})$$

Example 9: Compute the determinant of the matrix 3x3:

$$A=\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

Solution : A= $\begin{bmatrix} \overset{+}{1} & \overset{-}{5} & \overset{+}{0} \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

det A= 1 * det $\begin{bmatrix} \mathbf{4} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{0} \end{bmatrix}$ **-5 * det** $\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ **+0 * det** $\begin{bmatrix} \mathbf{2} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$ **= 1*(0-2)-5*(0-0)+0=-2**

$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$ **صيغة اخرى عند استخدام الاعمدة**

$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$ **صيغة اخرى عند استخدام الصفوف**

HW: **Compute the determinants:**

1. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$ **2.** $\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 2 & -2 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{vmatrix}$ **4.** $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 4 & 2 \end{vmatrix}$

If A is a triangular matrixمصفوفة مثلثية , then det A is the product of the entries on the main diagonal of A.

Determinant of triangle matrix =حاصل ضرب القيم الموجودة في المحور فقط لان عند استخدام الطريقة الاعتيادية فان الكثير من الحدود ستذهب مع الصفر

Example 10: find the determinant of upper triangle matrix

1. Upper triangle matrix A= $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ 2. Lower triangle matrix A= $\begin{bmatrix} 5 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 7 & 1 \end{bmatrix}$

3. Diagonal matrixالمصفوفة القطرية A= $\begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

Solution :

1. **det A= 2 det** $\begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}$ **-3 det** $\begin{bmatrix} 0 & 5 \\ 0 & 6 \end{bmatrix}$ **+1 det** $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ **= 2*(4*6-5*0)-3*(0*6-5*0)+1(0*0-2*0)=2*4*6-0+0=2*4*6=48 للتوضيح**
2. det(A)=5*3*1=15 حل مباشر
3. det(A)=7*2*9=126

Inverse Matrices (Nonsingular) المصفوفات العكسية

An n x n matrix A is said to be invertible انعكاسية if there is an n x n matrix C such that:

$$CA = I \text{ and } AC = I$$

where $I = I_n$= **identity** matrix. In this case, C is an inverse of A.

$A^{-1}A = I$ and $AA^{-1} = I$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 11: check if the C is inverse of A

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \text{ and } C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

Solution :

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \text{Thus } C = A^{-1}$$

Invertible matrix called **nonsingular matrix**مصفوفة غير منفردة .

Not invertible matrix called **singular matrix**.مصفوفة منفردة

القاعدة تنص على ان ارقام الطرفين تتبادلان وارقام الوسطين تعكس اشارتهما ثم تقسم على قيمة detالخاص بالمصفوفة

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

Example 12: Find the inverse of : $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

Solution : Since det A = 3(6)- 4(5) =- 2 ≠ 0, A is invertible, and

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 6/(-2) & -4/(-2) \\ -5/(-2) & 3/(-2) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

Solve system of equations by using inverse matrix حل المعادلات باستخدام معكوس المصفوفة

If A is an invertible انعكاسية n x n matrix, then solve of the equation **Ax = b** has the unique solution **x = A⁻¹b** , where $x=x_1 , x_2$.

Ax = b → x = A^{-1}b

Example 13: solve the system of equations by using inverse matrix

$$3x_1 + 4x_2 = 3$$
$$5x_1 + 6x_2 = 7$$

Solution :

$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ → Ax = b → x=A^{-1}b

Calculate A^{-1} (inverse of A)

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

det A= ad-bc=3*6-4*5=18-20=-2 → $A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$ حاصل ضرب رقم مع مصفوفة

$A^{-1} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$, then $x=A^{-1}b=\begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ حاصل ضرب صف مع عمود

$x=\begin{bmatrix} -3 * 3 + 2 * 7 \\ \frac{5}{2} * 3 + \left(-\frac{3}{2}\right) * 7 \end{bmatrix}=\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ → $x_1=5$, $x_2=-3$

HW

Find the inverses of the matrices

1. $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$ **2.** $\begin{bmatrix} 5 & 4 \\ 9 & 7 \end{bmatrix}$

3. $\begin{bmatrix} 8 & 3 \\ -7 & -3 \end{bmatrix}$ **4.** $\begin{bmatrix} 3 & -2 \\ 7 & -4 \end{bmatrix}$

Use the inverse matrix to solve the system

$$\begin{aligned} 5x_1 + 4x_2 &= -3 \\ 9x_1 + 7x_2 &= -5 \end{aligned}$$