Ordinary Differential Equations

An ordinary differential equation (ODE) is an equation involving an unknown function of one variable and some its derivatives, while a partial differntial equation (PDE) can be defined as is an equation involving an unkown function of two or more variables and certian of its partial derivatives.

Examples

1- The equation

$$\frac{du}{dt} = y^2, \tag{2.1}$$

where $u : R \longrightarrow R$, is an ODE .

2- The equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 y}{\partial x^2},$$

where $u : R^2 \longrightarrow R$, is a PDE.

Remark 2.1. in the ODEs we may refer for simplisity $\frac{dy}{dt} = y_t$ or y', therefore equation (2.1) can be rewrittin in this way

$$y' = y^2.$$

Definition 2.2. The order of any differential equation is the order highest derivative which appears in the equation.

Definition 2.3. For any diffferential equation, we say that it is linear when it is linear with respect to the dependent variable y, otherwise we say that the equation is nonlinear.

Examples

1-

$$y'' + y' + y = \sin x, \qquad \text{is a second order linear ODE.}$$

2-

$$y' + y^2 = 0, \qquad \text{is a first order nonlinear ODE}$$

3-

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \qquad \text{is a second order linear PDE}$$

Definition 2.4. The function $y = y(t)$, is called is a solution to a ODE on the open interval I, if it satisfies the equation and defined on I.

Example

it is easy to see that the function

$$y = \frac{1}{(c - t)}, \tag{2.2}$$

is defind on $R/\{c\}$, where $c \in R$,

and satisfy of the following ODE

$$y' = y^2. \tag{2.3}$$

Therefore, it is a solution to this ODE on $R/\{c\}$.

Exercises

1- For each of the following differential equations study the type (ODE or PDE), (Linear or nonlinear), and show the order.

(i) $y' = \sin(y) + t$

(ii) $y_t = y_x + e^{t+x}$

(iii) $\cos(y'') = t^2$

(iv) $y'' + y = \tan(t)$.

3 Methods for Solving First Order Equations

We will study some mothds used to find the solutions of the first order equations which take the form

$$y' = f(y, t).$$

1- Separable Equations

Finding a way to separate the variables is almost always the best method to attempt first when trying to solve a differential equation. Even if one of the methods that we will discuss later works for a given differential equation, we will invariably end up with the same integral to solve. Formally, a differential equation is separable if it can be written in the form

$$\frac{dy}{dt} = f(y,t) = a(t)b(y)$$

where $a, b : R \longrightarrow R$ are continuous functions
and the solution is

$$\int \frac{dy}{b(y)} = \int a(t)dt.$$

It is not always easy to determine whether or not a given differential equation is separable. The following theorem addresses this problem.

Theorem 3.1. *The differential equation $y' = f(y,t)$, is separable if and only if*

$$f(t,y)\frac{\partial^2 f}{\partial t \partial y} = \frac{\partial f}{\partial t}\frac{\partial f}{\partial y}.$$

Example
Determine if $y' = 1 + t^2 + y^3 + t^2y^3$, is separable

Setting $f(t,y) = 1 + t^2 + y^3 + t^2y^3$ and taking the necessary partial derivatives,

$$\frac{\partial f}{\partial t} = 2t + 2ty^3,$$

$$\frac{\partial f}{\partial y} = 3y^2 + 3t^2y^2,$$

Hence

$$\frac{\partial f}{\partial t}\frac{\partial f}{\partial y} = 6ty^2 + 6t^3y^2 + 6ty^5 + 6t^3y^5.$$

and

$$f(t,y)\frac{\partial^2 f}{\partial t \partial y} = 6ty^2 + 6t^3y^2 + 6ty^5 + 6t^3y^5 = \frac{\partial f}{\partial t}\frac{\partial f}{\partial y}$$

Exersise:Find the solution of the following equation

$$y' = y\sin(t).$$

2- Homogeneous Equations

An ordinary differential equation is said to be a homogeneous differential equation if the following condition is satisfied

$$y' = f(zt, zy) = f(t, y),$$

for any $z \in R$.

Set $y = vt$, thus the general form of first order ODE becomes

$$y' = \frac{dy}{dt} = \frac{dt(vt)}{dt} = v + t\frac{dv}{dt} = f(t, vt).$$

Since this equation is homogenous, we can use separation of variables to solve the equation

$$v' = \frac{f(t, vt) - v}{t}.$$

Example

Find the solution of the following equation

$$y' = \frac{y^2 + 2ty}{t^2}.$$

set

$$f(y, t) = \frac{y^2 + 2ty}{t^2}$$

Clearly,

$$f(zt, zy) = \frac{(zy)^2 + 2(zt)(zy)}{(zt)^2} = f(t, y).$$

Therefore, this equation is homogenuos

Now to find the solution, we set $y = vt$, and the equation can be written as follows

$$v' = \frac{\frac{v^2t^2+2tvt}{t^2} - v}{t} = \frac{v^2 - v}{t}$$

Thus

$$\frac{dt}{t} = \frac{dv}{v^2 + v}$$

if you integrate the two sides, we get

$$\ln(t) = \int \frac{dv}{v(v+1)} = \int (\frac{A}{v} + \frac{B}{v+1})dt$$

It is not difficult to see that $A = 1, B = -1$, thus the last equation becomes

$$\ln(t) = \ln(v) - \ln(v+1) + c = \ln(\frac{v}{v+1}) + c.$$

Thus

$$t = \frac{v}{(v+1)}e^c,$$

set $e^c = k$, we get

$$t = k\frac{v}{v+1}.$$

Thus

$$tv + t = kv.$$

i.e. $v(t-k) = -t$, thus

$$v = \frac{y}{t} = \frac{t}{k-t}$$

Therefore,

$$y = \frac{t^2}{(k-t)}.$$

as the general solution of the original differential equation.

Exercise Find the general solution of $y' = (y/t) - 1$.

3- Exact Equations

Consider the differential equation which takes the form

$$M(t,y)dt + N(t,y)dy = 0,$$

we say that this differntial equation is exact if it catisfied this condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}.$$

To solve an Exact Equation $M(t,y)dt+N(t,y)dy = 0$, we have to follow the following steps

(i) Assume that the function ϕ is a for t and y (the solution of the general equation), such that

(ii) Set $M(t,y) = \frac{\partial \phi}{\partial t}$, $N(t,y) = \frac{\partial \phi}{\partial y}$

(iii) Integrate $M(t,y) = M(t,y) = \frac{\partial \phi}{\partial t}$ in ti t to optain

$$\phi(t,y) = \int_t M(s,y)ds + h(y)$$

(iv) Calculate $\frac{\partial \phi}{\partial y}$ from the expression for $\phi(t,y)$ in step 2. The solution is $\phi(t,y) = C$, where C is a constant.

(v) Set the expression for $\frac{\partial \phi}{\partial y}$ obtained in step (3) equal to $N(t,y)$. This should give a differential equation for $h(y)$.

(vi) Solve for $h(y)$.

(vii) Substitute the expression for $h(y)$ into the expression for $\phi(t,y)$ in step (2). The solution is $\phi(t,y) = C$, where C is aconstant.

Example Find the solution of the following differential equation

$$y' = -\frac{y\cos(t) + 2te^y}{\sin(t) + t^2e^y + 2}.$$

We can rewrite the differential equation as

$$(y\cos(t) + 2te^y)dt = (\sin(t) + t^2e^y + 2)dy$$

which has the form $M(t, y)dt + N(t, y)dy = 0$, where

$$M(t, y) = (y\cos(t) + 2te^y), \quad N(t, y) = (\sin(t) + t^2e^y + 2).$$

It is clear that

$$\frac{\partial M}{\partial y} = \cos(t) + 2te^y = \frac{\partial N}{\partial t}.$$

Assume that the function ϕ is a for t and the solution y of the general equation such that

$$\frac{\partial \phi}{\partial t} = M(t, y) = y\cot(t) + 2te^y, \tag{3.1}$$

$$\frac{\partial \phi}{\partial y} = N(t, y) = \sin(t) + t^2e^y + 2. \tag{3.2}$$

Integrate equation (3.1) over t, it follows that

$$\phi(t, y) = \int (y\cot(t) + 2te^y)dt = y\sin(t) + t^2e^y + h(y), \tag{3.3}$$

where h is an unknown function of y.

Differentiating the last equation with respect to y and setting the result equal to (3.2) gives

$$\frac{\partial \phi}{\partial y} = \sin(t) + t^2e^y + 2 = \sin(t) + t^2e^y + h'(y),$$

Canceling common terms of both sides of the equation gives $h'(y) = 2$ or $dh = 2dy$, which leads to

$$h(y) = 2y + c$$

Thus equation (3.3) becomes

$$\phi(t, y) = y\sin(t) + t^2e^y + 2y + c,$$

Therefore, if we consider $c = 0$, the family for the solution of the general equation takes the form

$$y\sin(t) + t^2e^y + 2y = C,$$