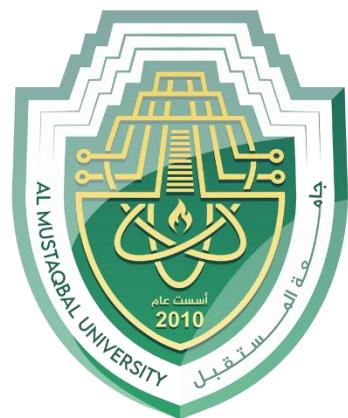**Al-Mustaqbal university**

Engineering technical college

Department of Building &Construction Engineering

Mathematics

First class

Lecture No.5

Assist. Lecture

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Chain Rule :

The chain rule is used to differentiate the composite functions.

1. Chain rule for function of single variable defined along paths.

$$y = f(x)$$
$$X = X(t) \implies \frac{dy}{dt} = \frac{dy}{dx} * \frac{dx}{dt}$$
$$y = F[X(t)]$$

Ex: if $y = X^2 + 1$, $X = \tan^{-1} t$, find $\frac{dy}{dt}$?

Sol: $\frac{dy}{dx} = 2X$

$$\frac{dx}{dt} = \frac{1}{1+t^2}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$= 2X \cdot \frac{1}{1+t^2} = 2\tan^{-1} t * \frac{1}{1+t^2}$$

2. For the function $Z = f(x,y)$ of two variable defined along path.

$$Z = f[x(t), g(t)]$$

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} + \frac{dy}{dt}$$

Ex: $Z = f(x,y) = x^2 y^3$, $x = \cos t$, $y = \sin t$ find $\frac{df}{dt}$?

Sol:

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}$$

$$f_x = y^3 2x = 2xy^3$$

$$f_y = 3y^2 x^2$$

$$\frac{dy}{dt} = \cos t \quad , \quad \frac{dx}{dt} = -\sin t$$

$$\frac{df}{dt} = 2xy^3 (-\sin t) + 3y^2 x^2 (\cos t)$$

$$= 2 \cos t (\sin t)^3 (-\sin t) + 3 (\sin t)^2 (\cos t)^2 (\cos t)$$

$$= -2 \cos t (\sin t)^4 + 3 (\sin t)^2 (\cos t)^3$$

3. chain rule for function to more than two variable defined along path :

$$W = f(x_1, x_2, x_3, \cdots, x_n)$$

$$\frac{df}{dt} = \frac{df}{dx_1} \cdot \frac{dx_1}{dt} + \frac{df}{dx_2} \cdot \frac{dx_2}{dt} + \cdots + \frac{df}{dx_n} \cdot \frac{dx_n}{dt}$$

Ex : let $W = xy \sin z$ where $x = \cos t$, $y = \sin t$, $z = 1 + t^2$

find $\frac{dw}{dt}$?

Sol :

$$\frac{df}{dt} = \frac{df}{dx_1} \cdot \frac{dx_1}{dt} + \frac{df}{dx_2} \cdot \frac{dx_2}{dt} + \frac{df}{dx_3} \cdot \frac{dx_3}{dt}$$

$$f_x = y \sin z, \quad f_y = x \sin z, \quad f_z = xy \cos z$$

$$\frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dt} = \cos t$$

$$\frac{dz}{dt} = 2t$$

$$\frac{dw}{dt} = y \sin z (-\sin t) + x \sin z (\cos t) + xy \cos z (2t)$$

$$= y \sin(1+t^2)(-\sin t) + \cos t \sin(1+t^2) x + \cos t \sin t \cos(1+t^2)$$
$$(2t)$$

4. Chain rule for function of two variable defined on surface :

If $z = f(x,y)$ has partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and if

$x = x(r,s)$ and $y = y(r,s)$

$z = f\left[x(r,s), y(r,s)\right]$

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Ex : find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$ if $z = f(x,y) = x^2 + y^2$

$x = r + e^s$, $y = \ln s$?

Sol :

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$f_x = 2x$, $\frac{\partial x}{\partial r} = 1$, $\frac{\partial y}{\partial r} = 0$

$f_y = 2y$, $\frac{\partial x}{\partial s} = e^s$, $\frac{\partial y}{\partial s} = \frac{1}{s}$

$$\frac{\partial z}{\partial r} = 2x(1) + 2y(0) = 2x = 2(r + e^s)$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= 2x\,e^{s} + 2y\,(1/s)$$

$$= 2\,(r+e^{s})\cdot e^{s} + 2\,(\ln s)\cdot \frac{1}{s}$$

$$= 2r\,e^{s} + 2\,e^{2s} + \frac{2}{s}\,\ln s$$

Ex: If z is a differentiable function of x and y which

Satisfy the equation $x^{3} + y^{3} + z^{3} + 3x^{2}\sin y\,\tan z = 5$?

find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$?

Sol:

$$3x^{2} + o + 3z^{2}\frac{\partial z}{\partial x} + 3\sin y\left(x^{2}\sec^{2}z\,\frac{\partial z}{\partial x} + \tan z * 2x\right) = 0$$

$$3x^{2} + 3z^{2}\frac{\partial z}{\partial x} + 3x^{2}\sin y\,\sec^{2}z\,\frac{\partial z}{\partial x} + 6x\,\sin y\,\tan z = 0$$

$$3x^{2}\sin y\,\sec^{2}z\,\frac{\partial z}{\partial x} + 3z^{2}\frac{\partial z}{\partial x} = -3x^{2} - 6x\,\sin y\,\tan z$$

$$\frac{\partial z}{\partial x}\left(3x^{2}\sin y\,\sec^{2}z + 3z^{2}\right) = -3x^{2} - 6x\,\sin y\,\tan z$$

$$\frac{\partial z}{\partial x} = \frac{-3x^{2} - 6x\,\sin y\,\tan z}{3x^{2}\sin y\,\sec^{2}z + 3z^{2}}$$

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$$0 + 3y^2 + 3z^2\frac{dz}{dy} + 3x^2\left(\sin y \sec^2 z \frac{dz}{dy} + \tan z \cos y\right) = 0$$

$$3y^2 + 3z^2\frac{dz}{dy} + 3x^2 \sin y \sec^2 z \frac{dz}{dy} + 3x^2 \tan z \cos y = 0$$

$$3z^2\frac{dz}{dy} + 3x^2 \sin y \sec^2 z \frac{dz}{dy} = -3y^2 - 3x^2 \tan z \cos y$$

$$\frac{dz}{dy}\left(3z^2 + 3x^2 \sin y \sec^2 z\right) = -3y^2 - 3x^2 \tan z \cos y$$

$$\frac{dz}{dy} = \frac{-3y^2 - 3x^2 \tan z \cos y}{3z^2 + 3x^2 \sin y \sec^2 z}$$

5. chain rule for function of three variables defined on surface :-

If $w = f(x, y, z)$ has partial derivatives

$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ and if $x = x(r,s)$, $z = z(r,s)$

also have partial derivatives

$$w = f\left[x(r,s), y(r,s), z(r,s)\right]$$

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$$

Ex: find $\frac{dw}{dr}$ if $w = f(x,y,z) = x+2y+z^2$ where $x = \frac{r}{5}$

$y = r^2 + e^s$, $z = 2r$?

Sol :

$$w = x + 2y + z^2$$

$$\frac{dw}{dx} = 1+0+0 = 1$$

$$\frac{dw}{dy} = 0+2+0 = 2$$

$$\frac{dw}{dz} = 0+0+2z = 2z$$

$$x = \frac{r}{5} \Rightarrow \frac{dx}{dr} = \frac{1}{5} * 1 = \frac{1}{5}$$

$$y = r^2 + e^s \Rightarrow \frac{dy}{dr} = 2r + 0 = 2r$$

$$z = 2r \Rightarrow \frac{dz}{dr} = 2$$

$$\frac{dw}{dr} = \frac{dw}{dx} \cdot \frac{dx}{dr} + \frac{dw}{dy} \cdot \frac{dy}{dr} + \frac{dw}{dz} \cdot \frac{dz}{dr}$$

$$= 1 * \frac{1}{5} + 2 * 2r + 2z * 2$$

$$= \frac{1}{5} + 4r + 4z$$

$$= \frac{1}{5} + 4r + 4(2r) = \frac{1}{5} + 12r$$