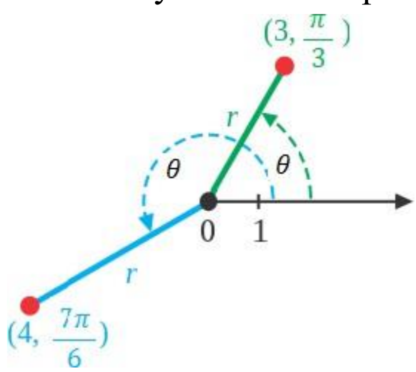
Polar Coordinates

In the Cartesian system the coordinates are perpendicular to one another with the same unit length on both axes.

A Polar coordinate system is determined by a fixed point, an origin, and a zero direction or axis. Each point is determined by an angle and a distance relative to the zero axis and the origin.

The distance from the origin is called the radial coordinate or radius, and the angle is called the polar angle.

The radial coordinate is often denoted by r and the polar angle by θ.



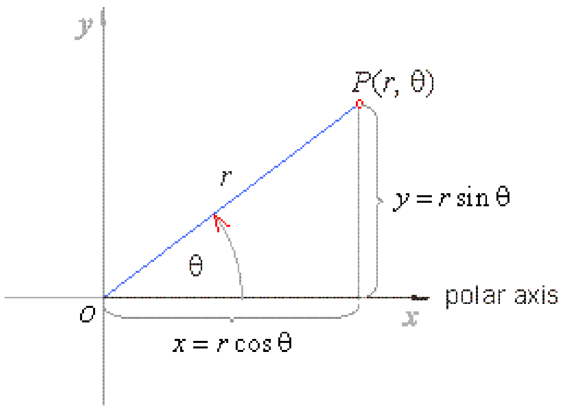
Converting between polar and Cartesian coordinates

The polar coordinates r and θ can be converted to the Cartesian coordinates x and y by using the trigonometric functions sine and cosine:

$x = r\cos\theta$ and $y = r\sin\theta$

The Cartesian coordinates x and y can be converted to polar coordinates r and θ by:

$r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$



Example 1: Convert each of the following points into the given coordinate system.

$(a)\ \ (-4, \frac{2\pi}{3})$ into Cartesian coordinates.

$(b)\ \ (-1, -\sqrt{3})$ into polar coordinates.

Solution

$$(a)\ \ x = r\cos\theta \ \Rightarrow\ x = -4\cos\frac{2\pi}{3} \ \Rightarrow\ x = -4\times(-\frac{1}{2}) = 2$$

$$y = r\sin\theta \ \Rightarrow\ x = -4\sin\frac{2\pi}{3} \ \Rightarrow\ x = -4\times(\frac{\sqrt{3}}{2}) = -2\sqrt{3}$$

$$(-4, \frac{2\pi}{3}) \equiv (2, -2\sqrt{3})$$

$$(b)\ \ r = \sqrt{x^2+y^2} \ \Rightarrow\ r = \sqrt{(-1)^2 + (-\sqrt{3})^2} \ \Rightarrow\ r = 2$$

$$\theta = \tan^{-1}\frac{y}{x} \ \Rightarrow\ \theta = \tan^{-1}\frac{-\sqrt{3}}{-1} \ \Rightarrow\ \theta = \pi + \frac{\pi}{3}$$

$$(-1, -\sqrt{3}) \equiv (2, \frac{4\pi}{3})$$

Example 2: Convert each of the following into an equation in the given coordinate system.

$(a)\ \ r = -8\cos\theta$ into Cartesian coordinates.

$(b)\ \ x^2 + y^2 - 2y = 2xy$ into polar coordinates.

Solution

$$(a)\ \ r = -8\cos\theta \ \Rightarrow\ r^2 = -8r\cos\theta$$

$$x^2 + y^2 = -8x$$

$$(b)\ \ x^2 + y^2 - 2y = 2xy$$

$$r^2 - 2r\sin\theta = 2\,r^2\ \sin\theta\cos\theta$$

$$r^2 - 2r\sin\theta = r^2\ \sin 2\theta$$

$$r^2 - r^2\ \sin 2\theta = 2r\sin\theta$$

$$r^2(1 - \sin 2\theta) = 2r\sin\theta$$

$$r = \frac{2\sin\theta}{1 - \sin 2\theta}$$

Double Integrals in Polar Coordinates

Let's suppose we wanted to do the following integral,

$$\iint_D f(x,y)\ dA \text{ , where } D \text{ is a disk of radius } a$$

To this we would have to determine a set of inequalities for x and y that describe this region. These would be

$$-a \le x \le a$$
$$-\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}$$

With these limits the integral would become,

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y) dydx$$

Now, if we're going to be converting an integral in Cartesian coordinates into an integral in polar coordinates we are going to have to make sure that we've also converted all the $x's$ and $y's$ into polar coordinates as well. To do this we'll need to remember the following conversion formulas,

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$

However, a disk of radius a can be defined in polar coordinates by the following inequalities, $0 \le \theta \le 2\pi$ and $0 \le r \le a$.

We can say that $dA = rdrd\theta$

So $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y) dydx = \int_{0}^{2\pi} \int_{0}^{a} f(r,\theta) rdrd\theta$

Let's look at a couple of examples of these kinds of integrals.

Example 3: Evaluate the integral by converting it into polar coordinates.

$$\iint_D 2xy \ dA \quad D = \{(r,\theta):\ 1 \le r \le 2\,, \qquad 0 \le \theta \le \pi/2\ \}$$

Solution:

$$\iint_D 2xy\, dA = \int_0^{\pi/2}\int_1^2 2(r\cos\theta)(r\sin\theta)r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_1^2 2\sin\theta\cos\theta\, r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2} \sin 2\theta \frac{r^4}{4}\Big|_1^2 \, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{16}{4} - \frac{1}{4}\right)\sin 2\theta\, d\theta$$

$$= -\frac{15}{8}\cos 2\theta\Big|_0^{\pi/2} = -\frac{15}{8}(\cos\pi - \cos 0) = \frac{15}{4}$$

Example 4: Evaluate $\iint_D e^{x^2+y^2} \ dA\,, D = \{(r,\theta): 0 \le r \le 1, \qquad 0 \le \theta \le 2\pi\ \}$

$$\iint_D e^{x^2+y^2}\ dA = \int_0^{2\pi}\int_0^1 e^{r^2} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}\, e^{r^2}\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(e-1)d\theta = \frac{1}{2}(e-1)\theta\Big|_0^{2\pi} = \pi(e-1)$$

Example 5: Evaluate $\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\,dx\,dy$, by converting it into polar coordinates.

$x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow r = 0 \quad or \quad \cos\theta = \sin\theta \Rightarrow \theta = \pi/4$

$x = \sqrt{2-y^2} \Rightarrow x^2 = 2 - y^2 \Rightarrow x^2 + y^2 = 2 \Rightarrow r = \sqrt{2}$

$y = 0 \Rightarrow r\sin\theta = 0 \Rightarrow \theta = 0$

$$\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\,dx\,dy = \int_0^{\pi/4} \int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^{\sqrt{2}} (\cos\theta + \sin\theta) r^2\,dr\,d\theta = \int_0^{\pi/4} (\cos\theta + \sin\theta) \left.\frac{r^3}{3}\right|_0^{\sqrt{2}} d\theta$$

$$= \int_0^{\pi/4} \frac{2\sqrt{2}}{3}(\cos\theta + \sin\theta)\,d\theta = \frac{2\sqrt{2}}{3}(\sin\theta - \cos\theta)\Big|_0^{\pi/4} = \frac{2\sqrt{2}}{3}$$

Exercises

1. Convert each of the following into an equation in the given coordinate system.

(a) $r = 4\cos\theta$ into Cartesian coordinates.

(b) $r = \sin 2\theta$ into Cartesian coordinates.

(c) $x^2 - 4x + y^2 = 0$ into polar coordinates.

(d) $3x + 2y = 4$ into polar coordinates.

2. Evaluate $\iint_D (x^2 + y^2)\,dx\,dy$, $D = \{(r,\theta): \quad 0 \le r \le \sqrt{3}, \quad 0 \le \theta \le \pi/2\}$

3. Evaluate $\iint_D (x + 3y)\,dA$, $D = \{(r,\theta): \quad 0 \le r \le 2, \quad 0 \le \theta \le \pi/4\}$