2.3 Exact First Order Differential Equation

$M(x,y) \cdot dx + N(x,y) \cdot dy = 0$

تكون المعادلة التفاضلية تامة (Exact) إذا توفر الشرط التالي:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If an equation can be written to the form:

$M(x,y) \cdot dx + N(x,y) \cdot dy = 0$

$$df(x,y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$M(x,y) = \frac{\partial f}{\partial x}, \quad \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$N(x,y) = \frac{\partial f}{\partial y}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

and have the property that:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ is said to be exact equation.

To find $f(x,y)$:

$$M(x,y) = \frac{df}{dx} \quad \rightarrow \quad \int df = \int M(x,y) \cdot dx$$

$$\therefore f(x,y) = \int M(x,y) \cdot dx + g(y)$$

$$\frac{df}{dy} = \int \frac{\partial M}{\partial y} \cdot dx + \frac{dg}{dy} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\because N = \frac{df}{dy} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Sub eq. (1) and eq. (2) to get:

$$\int \frac{\partial M}{\partial y} \cdot dx + \frac{dg}{dy} = N \quad \rightarrow$$

$$\frac{dg}{dy} = N - \int \frac{\partial M}{\partial y} \cdot dx$$

Then integral ∂g to find g(y) to get $f(x,y)$.

Example (1): Show that $y \cdot dx + x \cdot dy = 0$ is exact equation?

Solve:

$M = y, \;\; N = x$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} = 1 \\ \frac{\partial N}{\partial x} = 1\end{aligned}\right\} \;\; \rightarrow \;\; \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$$

$\therefore \; Exact\ function$

Example (2): Show that the equation $(x^2 + y^2) \cdot dx + (2xy + \cos y) \cdot dy = 0$ is exact equation and then find $f(x, y)$?

Solve:

$M = (x^2 + y^2), \;\; N = (2xy + \cos y)$

$$\left.\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2y\right\} \;\; \rightarrow \;\; \because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2y$$

$\therefore \; Exact\ function$

To find $f(x, y)$:

$$M = \frac{df}{dx} = x^2 + y^2 \;\; \rightarrow \;\; \int df = \int (x^2 + y^2) \cdot dx$$

$$\therefore f(x, y) = \frac{x^3}{3} + xy^2 + g(y)$$

$$\frac{df}{dy} = 2xy + \frac{dg}{dy} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

$$\because N = \frac{df}{dy} = 2xy + \cos y \ldots \ldots \ldots \ldots \ldots \ldots (2)$$

Sub eq. (1) and eq. (2) to get:

$$2xy + \frac{dg}{dy} = 2xy + \cos y \;\; \rightarrow$$

$$\frac{dg}{dy} = \cos y \;\; \rightarrow \;\; \int dg = \int \cos y \cdot dy + c$$

$$g(y) = \sin y + c$$

$$\therefore f(x, y) = \frac{x^3}{3} + xy^2 + \sin y + c$$