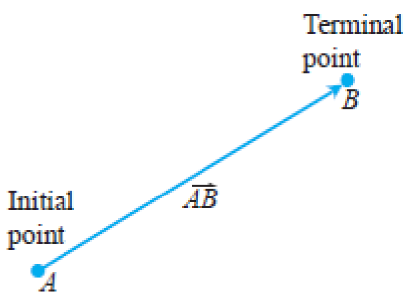
Vectors

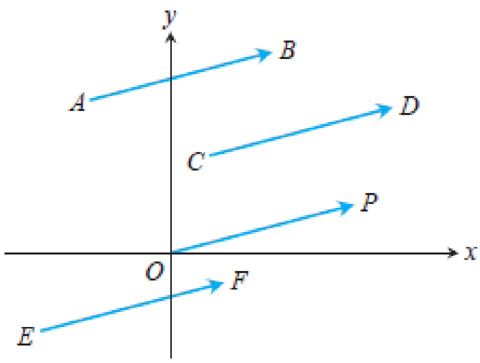
Definition of a vector

Mass, length, and time are examples of thinks, determined by their magnitudes. Whereas, force, velocity, and displacement, determined by their magnitudes and record their directions in which they act. These quantities are called ***vectors***.



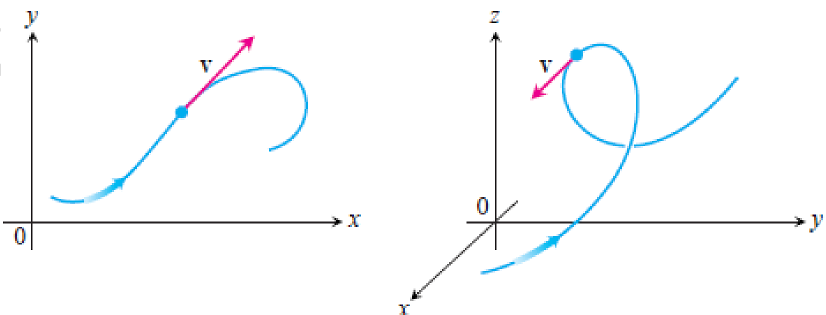
A **vector** in the plane is a directed line segment $\overrightarrow{AB}$ has initial point **A** and terminal point **B.**

The length is denoted by $|\overrightarrow{AB}|$.



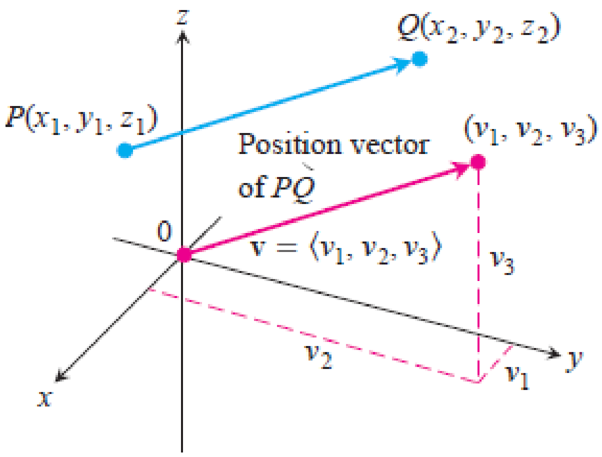
Two vectors are equal if they have the same length and direction. For example, $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$

The velocity vector of a particle moving along a path **(a) in the plane (b) in space**. The arrowhead on the path indicates the direction of motion of the particle.



(a) two dimensions (b) three dimensions

Component Form



If **v** is a two-dimensional vector in the plane equal to the vector with initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$ then the **component form** of v is: $\mathbf{v} = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$.

If **v** is a three-dimensional vector equal to the vector with initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$ then the **component form** of v is: $\mathbf{v} = \langle \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \rangle$. ; The numbers $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are called the **components** of **v** or $\overrightarrow{PQ}$.

$\mathbf{v_1}$: displacement in x – direction; $(\mathbf{x_1 + v_1 = x_2})$ and $\mathbf{v_1} = x_2 - x_1$

$\mathbf{v_2}$: displacement in y – direction; $(\mathbf{y_1 + v_2 = y_2})$ and $\mathbf{v_2} = y_2 - y_1$

$\mathbf{v_3}$: displacement in z – direction; $(\mathbf{z_1 + v_3 = z_2})$ and $\mathbf{v_3} = z_2 - z_1$

The magnitude or length of a vector.

The magnitude or length of the vector **v** $= \overrightarrow{PQ}$ is the nonnegative number:

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example: Find the component form and the length of the vector with an initial point P (-3 ,4 ,1) and terminal point Q (-5 ,2 ,2)?

Solution:

$P(x_1, y_1, z_1) = P(-3, 4, 1)$ and $Q(x_2, y_2, z_2) = Q(-5, 2, 2)$

$\mathbf{v_1} = x_2 - x_1 = -5 - (-3) = -2$; $\mathbf{v_2} = y_2 - y_1 = 2 - 4 = -2$; $\mathbf{v_3} = z_2 - z_1 = 2 - 1 = 1$

$\rightarrow v = \langle -2, -2, 1 \rangle$.

$|v| = \sqrt{(-2)^2 + (-2)^2 + 1^2} = 3$ unit length.

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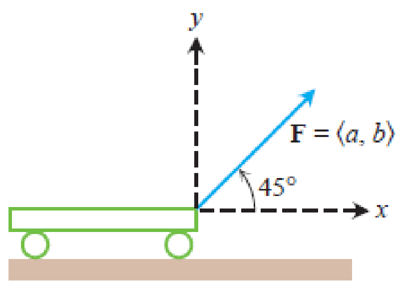
Example: A small cart is pulled along a smooth horizontal floor with a 20-lb force **F** making a 45° angle to the floor. What is the effective force moving the cart forward?



Solution:

The effective force is the horizontal component of **F** = ‹**a, b**› given by

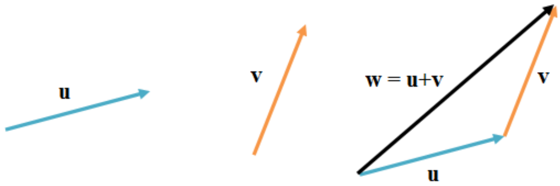
$$\boldsymbol{a = |F| \cos 45^o = (20)\left(\frac{\sqrt{2}}{2}\right) \approx 14.14 \text{ lb}}$$

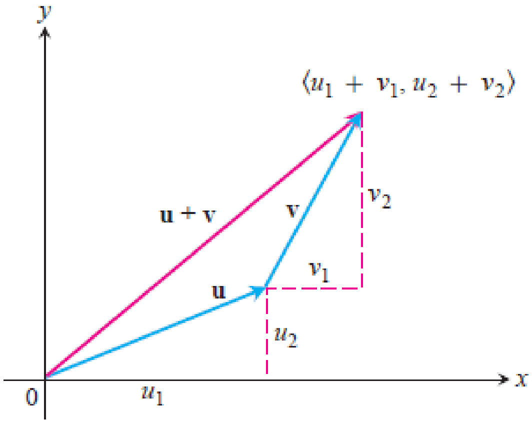
Vector Addition, Subtraction and Multiplication of a Vector.

Let vectors u = ‹ u_1, u_2, u_3› and v = ‹v_1, v_2, v_3 ›.

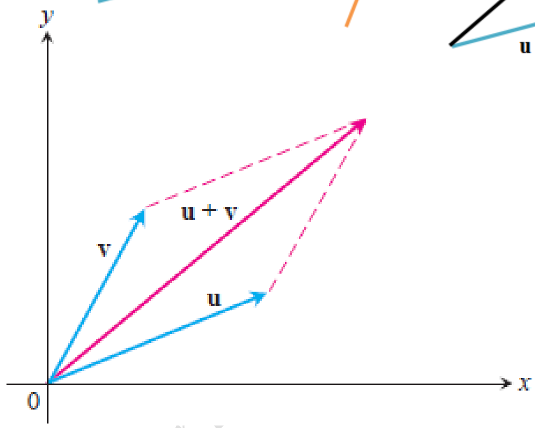
1. Vector Addition:

u + v = ‹$u_1 + v_1$, $u_2 + v_2$, $u_3 + v_3$›





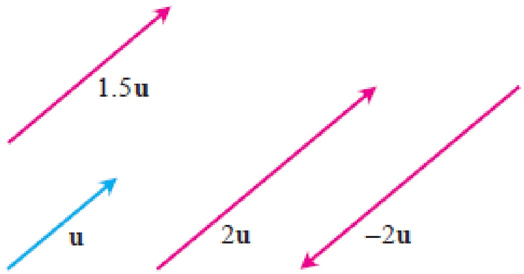
(a) Geometric interpretation of the vector sum,



(b) The parallelogram law of vector addition.

2. Multiplication of a Vector:

Ku = ‹ku_1, ku_2, ku_3›, k: scalar (number.)



$$|ku| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2}$$

$$= \sqrt{k^2({u_1}^2 + {u_2}^2 + {u_3}^2)} = \sqrt{k^2}\sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2} = |k||\boldsymbol{u}|$$

Example: Let u = ‹ -1, 3, 1› and v = ‹ 4, 7, 0›. Find

a) 2u + 3v, b) u – v, and c) $\left|\frac{1}{2}\text{u}\right|$?

Solution:

a) 2u+3v = 2 ‹ -1, 3, 1› + 3 ‹ 4, 7, 0› = ‹-2, 6, 2› + ‹12, 21, 0› = ‹10, 27, 1›.

b) u – v = ‹ -1, 3, 1› - ‹ 4, 7, 0› = ‹ -5, -4, 1›.

c) $\frac{1}{2}\text{u} = \frac{1}{2}\langle -1, 3, 1\rangle = \langle \frac{-1}{2}, \frac{3}{2}, \frac{1}{2}\rangle$.

$$\left|\frac{1}{2}\text{u}\right| = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11} \text{ unit length.}$$

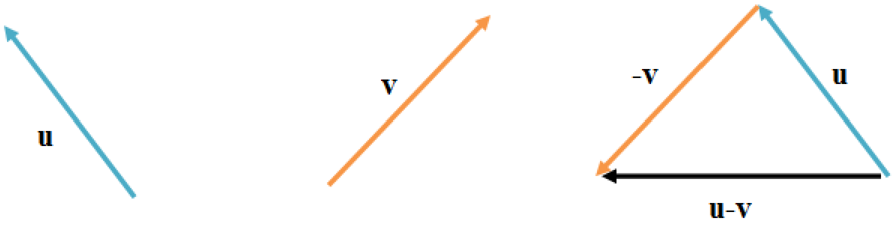
Properties of Vector Operations.

Let **u, v, w** be vectors and **a, b** be scalars.

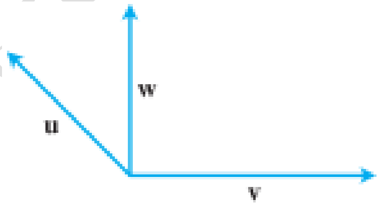
1. u + v = v + u
2. u + (v + w) = (u + v) + w
3. u + (-u) = 0 (zero vector)
4. u + 0 = u
5. 0u = 0
6. 1u = u
7. a (bu) = ab u
8. a (u + v) = au + av
9. (a + b) u = au + bu

3. Subtraction of vectors.

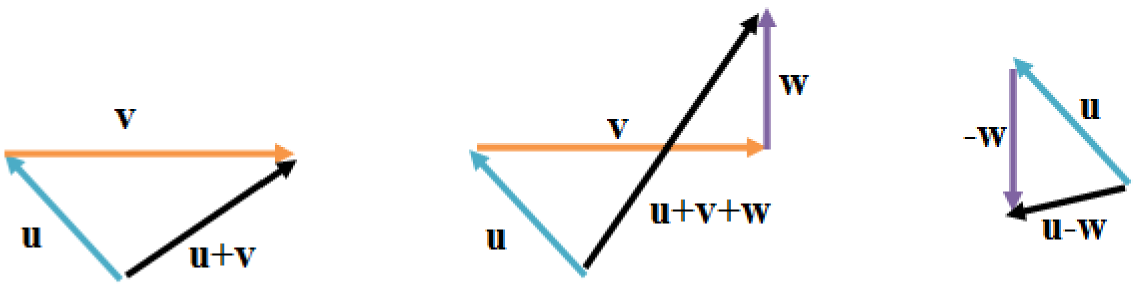
For any two vectors u and v, the difference u – v = u + (-v).



Example: For the following vectors, find u+v, u + v + w, u – w?



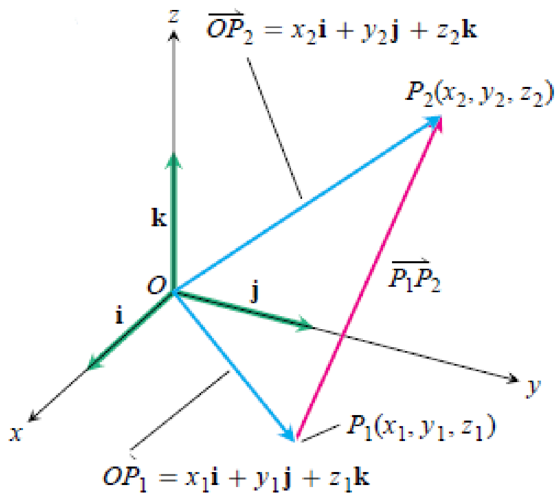
Solution:



Unit Vectors.

A vector **v** of length 1 is called a unit vector. The standard unit vectors are:

$i = \langle 1,0,0 \rangle,\qquad j = \langle 0,1,0 \rangle,\ \ k = \langle 0,0,1 \rangle$



The vector from p_1 to p_2 is:

$\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k.$

Any vector $v = \langle v_1, v_2, v_3 \rangle$ can be written as a linear combination of the standard unit vectors as: $v = v_1 i + v_2 j + v_3 k$

Unit vector of v = $\frac{v}{|v|}$, $|v| \neq 0$

Example: Find the **unit vector** of u from A (1, 0, 1) to B (3, 2, 0)?

Solution:

$u = (3-1)i + (2-0)j + (0-1)k = 2i + 2j - k$

$|u| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$ unit length

Unit vector of u = $\frac{u}{|u|} = \frac{2}{3}\, i + \frac{2}{3}\, j - \frac{1}{3}\, k$

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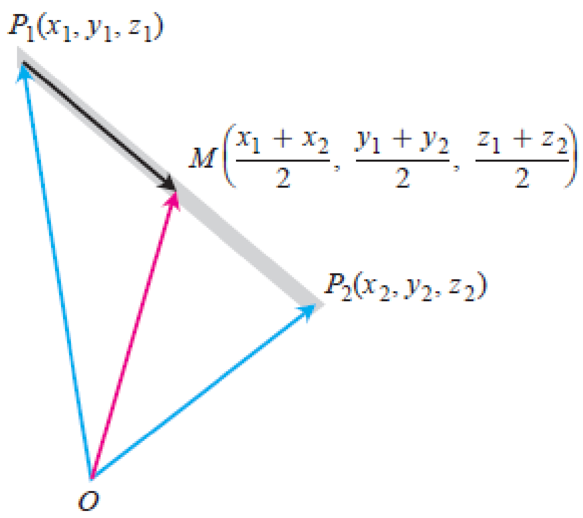
Example: Find the unit vector of v = *3i – 4j*?

Solution:

$|v| = \sqrt{3^2 + (-4)^2} = 5$ unit length

Unit vector of v = $\frac{v}{|v|} = \frac{3}{5} i - \frac{4}{5}\, j$

Midpoint of a Line Segment



The midpoint M of line segment joining $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point: $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

Example: Find the midpoint of the line segment joining P_1 (3 ,-2 ,0) and P_2 (7 ,4 ,0)?

Solution: $M\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+0}{2}\right) \rightarrow M(5, 1, 0)$.

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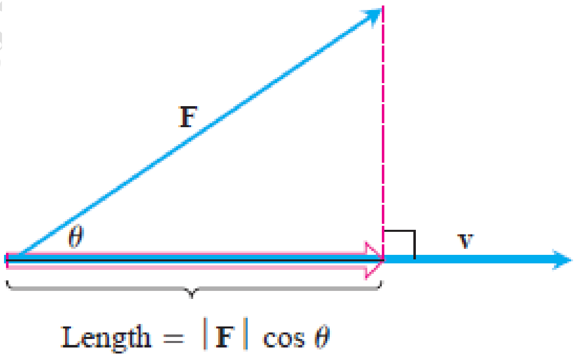
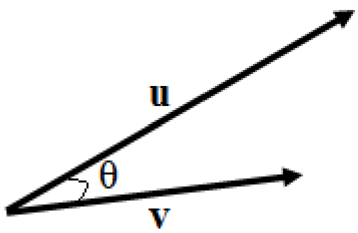
The Dot Product.



If a force **F** is applied to a particle moving along a path, we need to know the magnitude of the force in the direction of motion. If v is parallel to the tangent line to the path at the point where F is applied, then we want the magnitude of F in the direction of v. The dot product of two vectors like force and displacement is:



$$u.v = |u||v| \cos\theta$$

Definition.

Dot products are called **inner or scalar** products because the product results in a scalar, not a vector.

The dot product of vectors $u = < u_1, u_2, u_3 >$ and $v = < v_1, v_2, v_3 >$ is:

$$u.v = u_1v_1 + u_2v_2 + u_3v_3$$

Ex: Find the dot product of the following vectors:

1) $\langle 1, -2, -1\rangle . \langle -6, 2, -3\rangle$

The dot product = (1) (-6) + (-2) (2) + (-1) (-3) = -6 + (-4) + 3 = -7

2) $< \frac{1}{2}$ i + 3 j + k>. < 4 i – j + 2 k>

The dot product = $(\frac{1}{2})$ (4) + (3) (-1) + (1) (2) = 2 – 3 + 2 = 1

<u>Angle between two vectors.</u>

The angle between two nonzero vectors u = < u_1, u_2, u_3 > and v = < v_1, v_2, v_3 > is:

$$\theta = \cos^{-1}\frac{u_1v_1 + u_2v_2 + u_3v_3}{|u||v|}$$

Ex: Find the angle between u = < i – 2j – 2k> and v = < 6i + 3j + 2k>?

Solution:

$u.v = (1)(6) + (-2)(3) + (-2)(2) = -4$

$|u| = \sqrt{1^2 + (-2)^2 + (-2)^2} = 3$ unit length.

$|v| = \sqrt{6^2 + 3^2 + 2^2} = 7$ unit length.

$\theta = \cos^{-1}\frac{-4}{(3)(7)} = 1.762$ rad.

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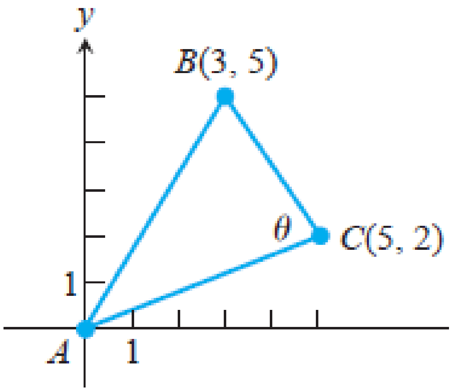
Ex: Find the angle of vertex C in the triangle ABC determined by the vertices A= (0,0) , B= (3,5) , C=(5,2)?



Solution:

$\overrightarrow{CA}$ = -5i – 2j

$\overrightarrow{CB} = -2i + 3j$

$\overrightarrow{CA}.\overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$

$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$ unit length.

$|\overrightarrow{CB}| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$ unit length.

$$\theta = \cos^{-1}\frac{4}{\sqrt{29}.\sqrt{13}} \approx 78.1^{\circ}$$

Orthogonal vectors.

Vectors u and v are orthogonal (or perpendicular) if and only if: u.v = 0

<u>Ex:</u> show that u and v are orthogonal if:

a) u = < -3, 2 >, v = < 4 , 6 >

u.v = (-3) (4) + (2) (6) = -12 + 12 = 0

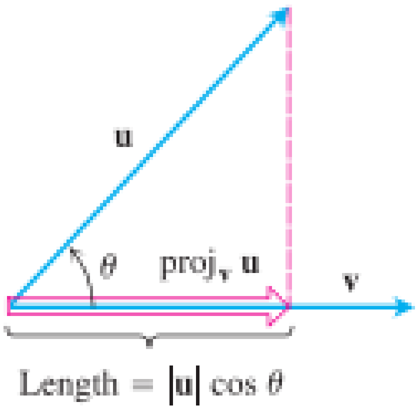
b) u = < 3i – 2j + k >, v = < 2j + 4k >

u.v = (3) (0) + (-2) (2) + (1) (4) = 0 – 4 + 4 = 0

<u>Vector projection and scalar component.</u>

The scalar component of **u** in the direction of **v** is:

$$|u|\cos\theta = \frac{u.v}{|v|}$$

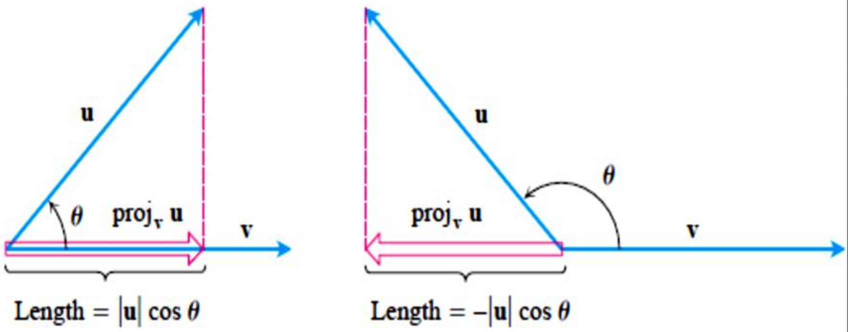


The vector projection of **u** onto **v** is:

$$Proj_v u = |u|\cos\theta \ .\frac{v}{|v|}$$

$$= \frac{u.v}{|v|}.\frac{v}{|v|}$$

$$= \frac{u.v}{|v|^2}\ v$$



(a) (b)

Ex: Find the vector projection of u = < 6i + 3j + 2k > onto v = < i – 2j – 2k> and the scalar component of u in the direction of v?

Solution:

u.v = (6) (1) + (3) (-2) + (2) (-2) = -4

$|v| = \sqrt{1^2 + (-2)^2 + (-2)^2} =$ 3 unit length.

$$\text{Proj}_v u = \frac{u.v}{|v|^2} \, .v = \frac{-4}{9}\ (i - 2j - 2k) = \frac{-4}{9}\ i + \frac{8}{9}\ j + \frac{8}{9}\ k$$

$$\text{Scalar component of u in the dir. of v} = \frac{u.v}{|v|} = \frac{-4}{3}$$

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Ex: Find the vector projection of F = 5i + 2j onto v = i – 3j?

Solution:

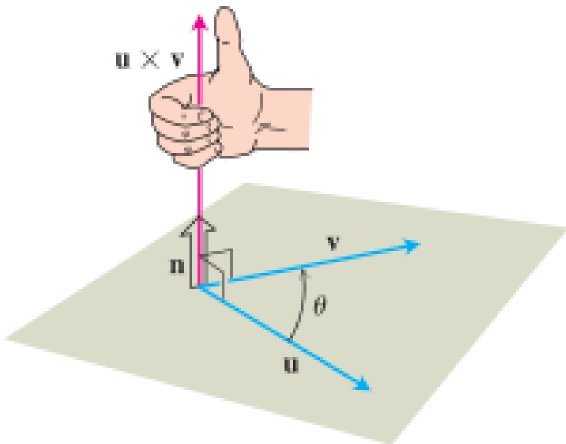
$$\text{Proj}_v F = \frac{(5)(1) + (2)(-3)}{\left(\sqrt{1^2 + (-3)^2}\right)^2}\ (i - 3j) = \frac{-1}{10} i + \frac{3}{10}\ j$$

Properties of the Dot Product.

If u, v, and w are any vectors and c is a scalar, then:

1) u . v = v . u
2) (c u).v = u. (c v) = c (u.v)
3) u. (v + w) = u . v + u . w
4) u. u = $|u|^2$
5) 0. u = 0

The Cross Product.



If **u** and **v** are two nonzero vectors, then the cross product **u × v** (u cross v) is:

$\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin\theta \, \vec{\mathbf{n}}$

Where **n** is a unit vector perpendicular on both u and v in the direction of u × v.

Unlike the dot product, the cross product is a vector. Therefore, it is called the vector product of u and v, and applies only to vectors in space.

Calculating the Cross Product as a Determinant

If $u = u_1 i + u_2 j + u_3 k$ and $v = v_1 i + v_2 j + v_3 k$, then

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Properties of the Cross Product.

If u, v, and w are any vectors and r, s are scalars, then:

1) u × v = -v × u
2) u × 0 = 0
3) (r u) × (s v) = (r s)(u × v)
4) u × (v + w) = u × v + u × w
5) u × (v × w) = (u. w) v – (u.v) w

Ex: Find u × v and v x u if u = 2i + j + k,   v = -4i + 3j + k?

Solution:

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$

= -2i – 6j + 10k, and

v x u = 2i + 6j – 10k

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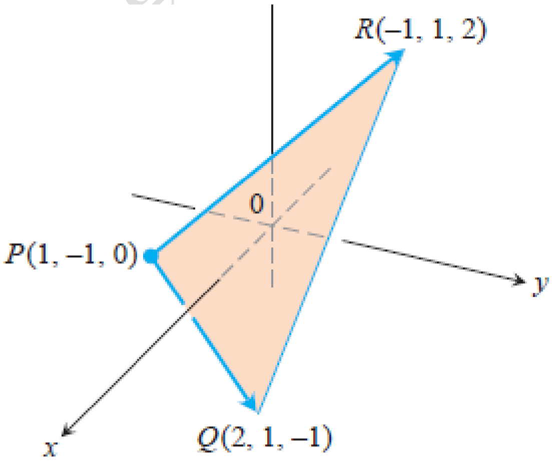
Ex: Find a vector perpendicular to the plane through P(1,-1,0), Q(2,1,-1), and R(-1,1,2) ?



Solution:

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular on both vectors.

$\overrightarrow{PQ} = i + 2j - k$

$\overrightarrow{PR} = -2i + 2j + 2k$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} k$$

= 6 i + 6 k

Ex: Find a unit vector perpendicular to the plane of P (1, -1, 0), Q (2, 1, -1), and R (-1, 1, 2)?

Solution: Since vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, its direction **n** is a unit vector perpendicular to the plane.

$$n = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6i + 6k}{6\sqrt{2}} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} k$$

Ex: Find a unit vector perpendicular to the plane ABC if A (2,-2, 1), B (3,-1, 2), and C (3,-1, 1)?

Solution:

Since $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to the plane, its direction **n** is a unit vector perpendicular to the plane.

$\overrightarrow{AB} = i + j + k$ & $\overrightarrow{AC} = i + j$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} j + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} k = -i + j$$

$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ unit length

$$\vec{n} = \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|} = \frac{-i+j}{\sqrt{2}} = \frac{-1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$$

Parallel Vectors.

Two nonzero vectors **u** and **v** are parallel if and only if **u × v = 0**.

Ex: Let u = 5i – j + k, w = -15i + 3j – 3k. Prove that the vector u and the vector w are parallel vectors?

Sol.

$$|u \times w| = \begin{vmatrix} i & j & k \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 3 & -3 \end{vmatrix} i - \begin{vmatrix} 5 & 1 \\ -15 & -3 \end{vmatrix} j + \begin{vmatrix} 5 & -1 \\ -15 & 3 \end{vmatrix} k$$

= 0 i + 0 j +0 k = 0 (u and w are parallel).