Subject Name: Control Systems2

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Lecture No. 1

Lecture Title: ROOT LOCUS

ROOT LOCUS ANALYSES:

A simple method for finding the roots of the characteristic equation has been developed by W. R. Evans and used extensively in control engineering. This method, called the *root-locus method,* is one in which the roots of the characteristic equation are plotted for all values of a system parameter. The roots corresponding to a particular value of this parameter can then be located on the resulting graph. Note that the parameter is usually the gain, but any other variable of the open-loop transfer function may be used.

By using the root-locus method the control Engineer can predict the effects on the location of the closed-loop poles with varying the gain value.

1. Root Locus Method

The root locus is the locus of roots of the characteristic equation of the closed-loop system as a specific parameter (usually, gain K) is varied from zero to infinity, giving the method its name. Such a plot clearly shows the contributions of each open-loop pole or zero to the locations of the closed-loop poles.
By using the root-locus method, it is possible to determine the value of the gain K that will make the damping ratio of the dominant closed-loop poles as prescribed.
If the location of an open-loop pole or zero is a system variable, then the root-locus method suggests the way to choose the location of an open-loop pole or zero.

- Root Locus Approach
  - Basic characteristic of the transient response of a closed-loop system is closely related to the location of the closed-loop poles.
  - If the system has a variable loop gain, then the location of the closed-loop poles depends on the value of the loop gain chosen.

- It is important to know how the closed-loop poles move in the *s* plane as the loop gain is varied.

- *The Root Locus Plot* is a plot of the roots of the characteristic equation of the *closed-loop system* for all values of a system parameter, usually the gain; however, any other variable (e.g., inertia, inductance, damping) of the open-loop transfer function may be used.

1.1 Angle and magnitude conditions

Consider the control system shown in Fig. 1, whose closed loop T.F. is;

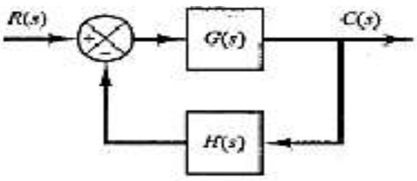


Fig.1 closed loop control system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The characteristic equation of this system is;

$$1 + G(s)H(s) = 0$$

The quantity G(s)H(s) is called loop T.F. or open-loop T.F. Assuming that the loop T.F. is a rational function including a gain K, this gives

$$1 + \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} = 0$$

Or $K\,G(s)H(s) = -1$ (1)

Since G(s)H(s) is a complex quantity, eqn. (1) can be split into two equations by equating the angles and magnitudes of both sides, respectively, to obtain the following:

Angle condition:

$$\underline{/G(s)H(s)} = \pm 180°(2k + 1) \qquad (k = 0, 1, 2, \ldots)$$

$\sum angle\ of\ zeros - \sum angle\ of\ poles = \pm 180$ (2)

Magnitude condition:

$|G(s)H(s)| = 1$ (3)

The values of s that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles.

Then the root loci for the system are the loci of the closed-loop poles as the gain K is varied from zero to infinity.

2. Root Locus Sketch

To begin sketching the root locus of a system by the root-locus method we must know the location of the poles and zeros of $G(s)\, H(s)$

■ **Step #1**

K= 0 points are located at the open-loop poles (number of poles = n)

■ **Step #2**

K= ∞ points are located at the open-loop zeros (number of zeros = m)

The poles and zeros referred above include those at infinity, if any.

Find the m zeros z_i and n poles p_j of $P(s)$

$$1 + K \frac{\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0.$$

- Locate the poles and zeros on the s-plane with selected symbols (o-zero, X-pole)
- The RL (root locus) starts at the n open-loop poles
- The RL ends at the open loop zeros, m of which are finite, n-m of which are at infinity

EX: Consider the characteristic equation

$$s(s+2)(s+3) + K(s+1) = 0$$

Dividing both sides by the terms that do not contain K, we get

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s+2)(s+3)} = 0$$

$$G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+3)}$$

When $K = 0$, the 3 poles are at $s = 0$, $s = -2$, and $s = -3$ as shown in Fig. 2.

$$\left.\begin{matrix} s = 0 \\ s = -2 \\ s = -3 \end{matrix}\right\} n = 3$$

When K is ∞, the 3 zeros are at $s = -1$, $s = \infty$ and ∞ as shown in Fig. 2.

$$\left.\begin{matrix} s = -1 \\ s \to \infty \\ s \to \infty \end{matrix}\right\} m = 1$$

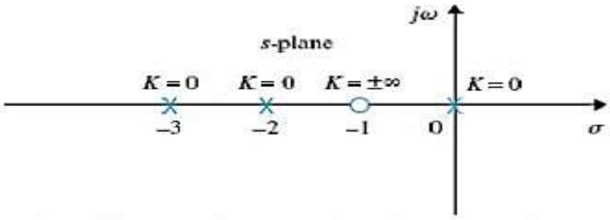


Fig. 2, K=0 and K=∞ points of the root locus

Number of branches on the root loci

We must know that, the number of branches of root locus plot equals the number of poles. In the previous example, there are 3 poles. So that the total number of root locus branches is **THREE**. Also for the control system that has 3 poles and shown in Fig. 3, it has three root locus branches.

Symmetry of The root loci

Root locus is symmetrical w.r.t. the real axis of the s-plane as shown in Fig. 3.

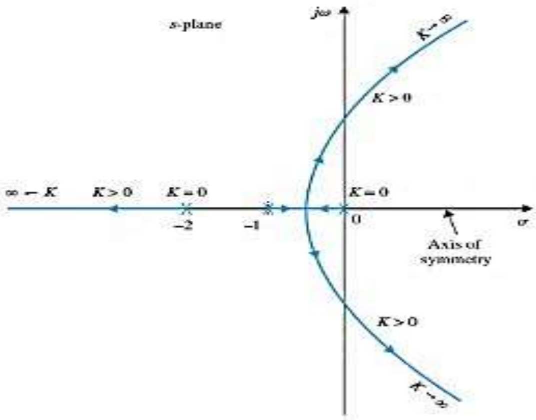


Fig. 3, three pole system gives three root locus branches; also the root locus is symmetrical around the real axis

■ Step #3

Number of Asymptotes if there are zeros located at ∞, there are asymptotes equal to those zeros at ∞. Simply we can calculate the number of asymptotes by:

$$\text{Number of asymptotes} = |n - m|$$

For the previous example, since there are **TWO** zeros at infinity OR $n - m = 2$, there are **TWO** asymptotes.

■ Step #4

Angle of Asymptotes we can calculate the angles of asymptotes by

$$\theta_k = \frac{(2k+1)180}{|n-m|}$$

Where $k = 0, 1, 2, \ldots, |n-m| - 1$

Substituting k=0 we get the angle of 1st asymptote

Substituting k=1 we get the angle of 2st asymptote, … etc.

In case of number of asymptotes =2, therefore the angles are $\theta_0 = 90$ and $\theta_1 = 270$ (as shown in Fig. 4-a)

In case of number of asymptotes =3, therefore the angles are $\theta_0 = 60$, $\theta_1 = 180$ and $\theta_2 = 300$ (as shown in Fig. 4-b)

In case of number of asymptotes =4, therefore the angles are $\theta_0 = 45$, $\theta_1 = 135$, $\theta_2 = 225$ and $\theta_3 = 315$ (as shown in Fig. 4-c)

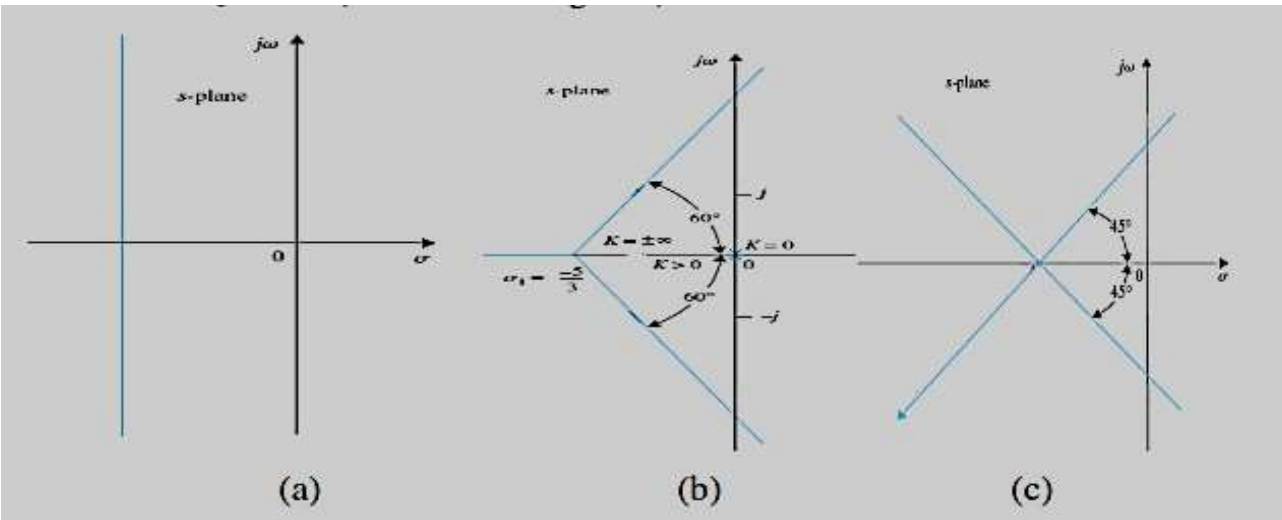


Fig. 4, angle of asymptotes

Step #5

Intersection of Asymptotes with Real Axis The point of intersection of asymptotes of the root locus lies on the real axis of the s-plane, at σ, where

<u>**Intersection of Asymptotes with Real Axis**</u> The point of intersection of asymptotes of the root locus lies on the real axis of the s-plane, at σ, where

$$\sigma = \frac{\sum finite\ poles\ of\ G(s)H(s) - \sum finite\ zeros\ of\ G(s)H(s)}{n-m}$$

The point of intersection of the asymptotes (σ) represents the center of gravity of the root locus, and is always a real number. Since the poles and zeros of $G(s)H(s)$ are either real or in complex-conjugate pairs, the imaginary parts in the numerator of σ equation always cancel each other out. Thus, the summation terms may be replaced by the real parts of the poles and zeros of $G(s)H(s)$, respectively. That is,

$$\sigma = \frac{\sum real\ part\ of\ poles\ of\ G(s)H(s) - \sum real\ part\ of\ zeros\ of\ G(s)H(s)}{n-m}$$

Example: suppose we have a control system

$$G(s)H(s) = \frac{K(s+1)}{s(s+4)(s^2+2s+2)}$$

$$G(s)H(s) = \frac{K(S+1)}{S(S+4)(S+1-j)(S+1+j)}$$

The point of intersection of asymptotes with real axis is

$$\sigma = \frac{(0-4-1-1)-(-1)}{4-1} = \frac{-5}{3} = -1.67$$

■ **Step #6**

Root Locus on Real Axis

On a given section of the real axis, root locus are found in this section only if the total number of poles and zeros of G(s)H(s) to the right of the section is ***odd***.

On another explanation, for $s1$ to be a point on the root locus, there must be an ***odd*** number of poles and zeros of $G(s)H(s)$ to the right of that point. We can explain this by the following pole-zero configurations shown in Fig. 5.

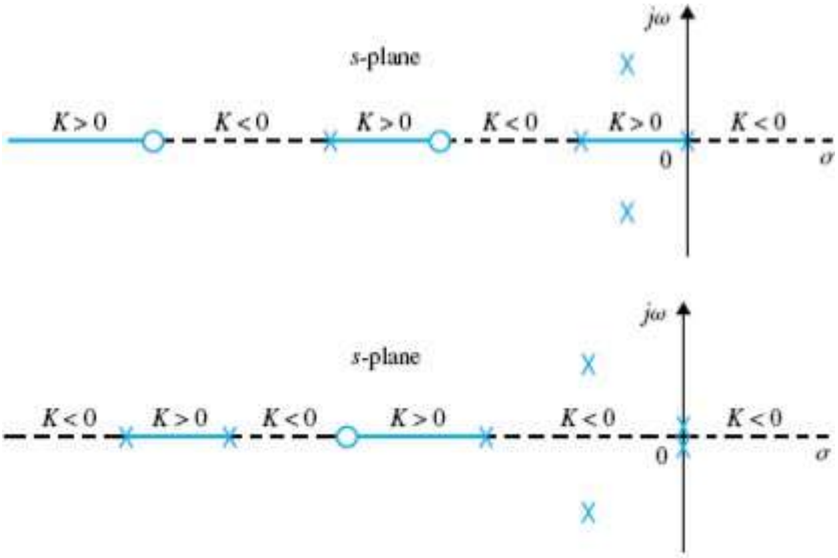


Fig. 5, root locus on real axis

Where the dotted line shows Inverse Root Locus (IRL) where the system gain K changes from $-\infty$ to 0 (i.e. K<0).

On the other hand, the solid line shows Root Locus (RL) where the system gain K changes from 0 to ∞ (i.e. K>0)

■ **Step #7**

Intersection of the Root Locus with the Imaginary Axis

The points where the root locus intersect the imaginary axis of the *s*-plane, and the corresponding values of *K*, may be determined by means of the Routh-Hurwitz criterion explained in the previous lecture.

■ Step #8

Breakaway Points

Breakaway points on the root locus of an equation correspond to multiple-order roots of the equation.

Figure (6-a) illustrates a case in which two branches of the root locus meet at the breakaway point on the real axis and then depart from the axis in opposite directions. In this case, the breakaway point represents a double root of the equation when the value of *K* is assigned the value corresponding to the point.

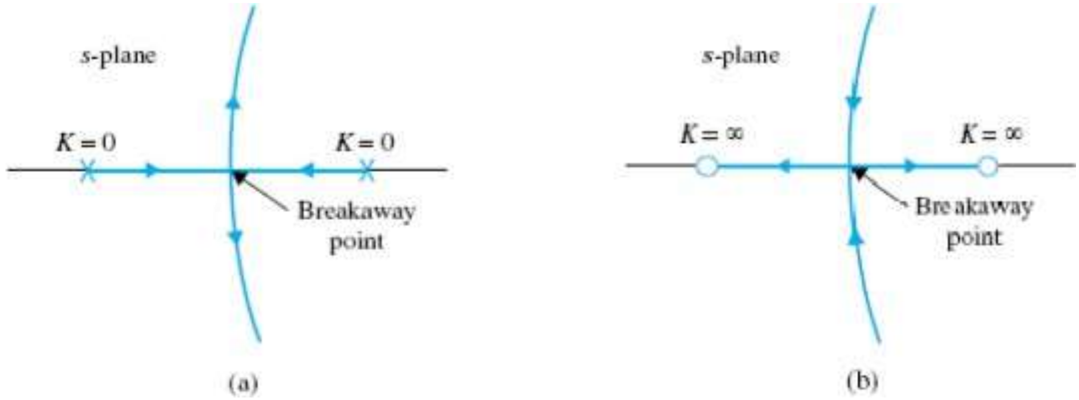


Figure (6-b) shows another common situation when two complex-conjugate root locus approach the real axis, meet at the breakaway point, and then depart in opposite directions along the real axis. In general, a breakaway point may involve more than two root locus.

Example1: Consider the second-order equation

- Obtaining the breakaway points

Rewriting the characteristic equation to isolate :

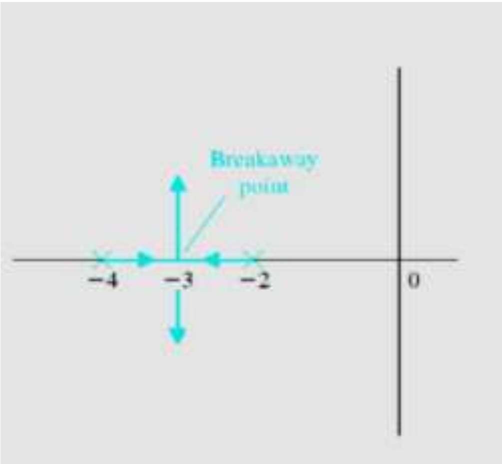
$$p(s) = K$$

The breakaway point occur when $\dfrac{dK}{ds} = \dfrac{dp(s)}{ds} = 0$

Example:

$$1 + \frac{K}{(s+2)(s+4)} = 0 \;\Rightarrow\; K = p(s) = -(s+2)(s+4)$$

$$\text{or } K = -(s^2 + 6s + 8) \;\Rightarrow\; \frac{dp(s)}{ds} = -(2s+6) = 0$$



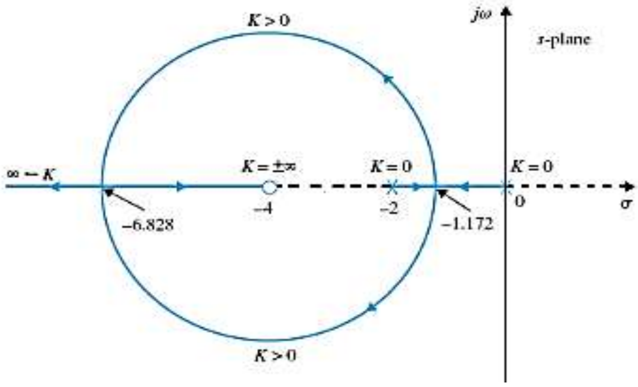
Example2: Consider the second-order equation

$$G_1(s)H_1(s) = \frac{s+4}{s(s+2)}$$

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{s(s+2) - 2(s+1)(s+4)}{s^2(s+2)^2} = 0$$

Or $s^2 + 8s + 8 = 0$

We find the two breakaway points of the root locus at $s = -1.172$ and -6.828



<u>Another example:</u> In this example, we shall show that not all the solutions of Eqn. (3) are breakaway points on the root locus. Consider the root loci of the equation

$$s(s^2 + 2s + 2) + K = 0$$

$$1 + KG_1(s)H_1(s) = 1 + \frac{K}{s(s^2 + 2s + 2)} = 0$$

$$3s^2 + 4s + 2 = 0$$

The roots of the above equation is $s_1 = -0.667 - j0.471$ and $s_2 = -0.667 + j0.471$. From the root locus shown in Fig. 11, these two roots are **not** breakaway points on the root loci.

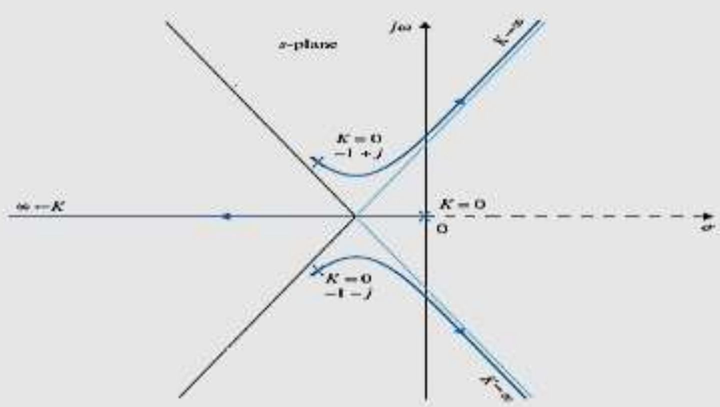


Fig. 11 Breakaway points of the example

■ Step #9

Angles of Departure for complex poles OR

And Angles of Arrival for complex zeros

The angle of departure or arrival of a root locus at a pole or zero, respectively, of G(s)H(s) denotes the angle of the tangent to the locus near the point. The angle of departure is defined as the angle at which the root locus leaves the pole. The angle of arrival is defined as the angle at which the root locus moves toward the zero.
We can explain how to calculate the angle of departure by the following example:
Consider the characteristic equation of a control system

$$S(S+3)(S^2+2S+2) + K(S+1) = 0$$

The angle of departure of the root locus at $(s+1-j)$ is represented by θ_2, measured with respect to the real axis. Let us assign s_1 to be a point on the RL leaving the

pole at $(s+1-j)$ and is very close to the pole as shown in Fig. 6. Then, $s1$ must satisfy Eqn. (2). Thus,

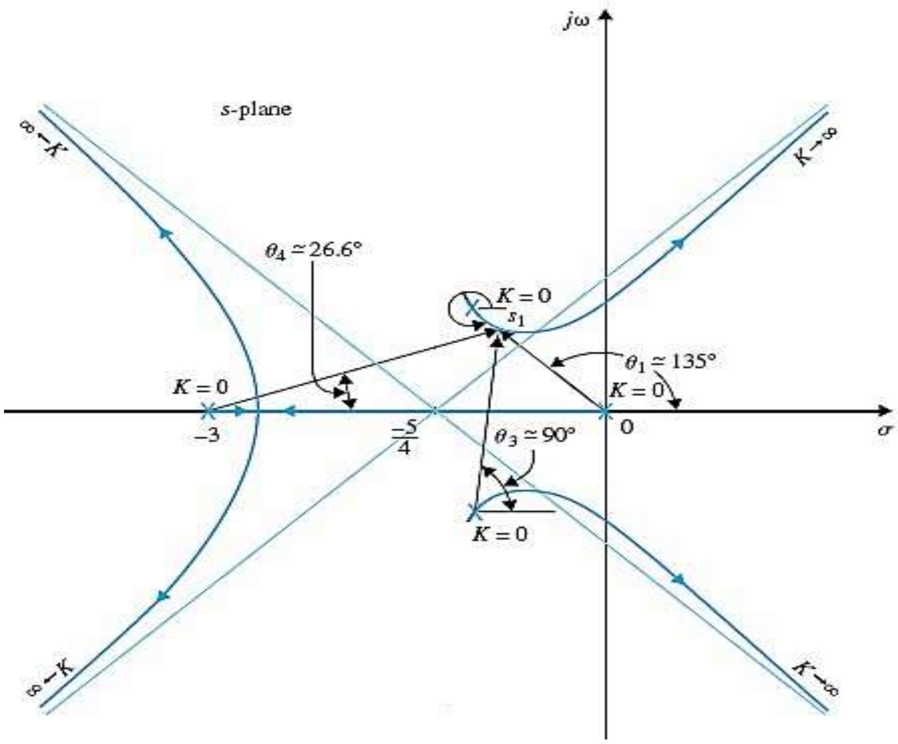


Fig. 6, calculation of angle of departure

$$\underline{/G(s)H(s)} = \pm 180°(2k+1) \qquad (k = 0, 1, 2, \ldots)$$

$$\sum angle\ of\ zeros - \sum angle\ of\ poles = \pm 180 \qquad (2)$$

Based on eqn. 2, since no zeros

$$0 - (\theta_1 + \theta_2 + \theta_3 + \theta_4) = 180$$

$$\theta_2 = -180 - \theta_1 - \theta_3 - \theta_4 = -180 - 135 - 90 - 26.6 = -71.6$$

■ Step #10

The final step of the root locus procedure are used to determine a root location s_1. and the value K_x at the root location.

- Determine the parameter value K_x at a specific root s_x using the magnitude requirement. The magnitude requirement s_x at is

$$K_x = \left. \frac{\prod_{j=1}^{n} |(s + p_j)|}{\prod_{i=1}^{M} |(s + z_i)|} \right|_{s=s_x}$$

The operating point s_1 for a given damping ratio, zeta (ζ) can be determined as follows:

First, determine the operating point angle, $\theta = \cos^{-1}(\zeta)$.
Then draw a line from the origin of the s-plane that forms an angle of θ with the negative real axis.
This line intersects the root locus at the operating point s_1.
The length of the line from the origin to the operating point represents the natural frequency, w_o.
The value of K at the operating point can be determined as follows:
First, draw lines from each pole and zero to the operating point.
Measure the length of each line and determine the product of the lengths of the lines from the poles and the product of the lengths of the lines from the zeros.
If there are no zeros (or poles) then use 1 as the product.
The value of K at the operating point is **equal** to the poles product divided by the zeros product.

Example 1 : Consider the following system , plot the root locus.

$$G(S)H(S) = \frac{K}{S(S+4)(S+4+j4)(S+4-j4)}$$

Step#1

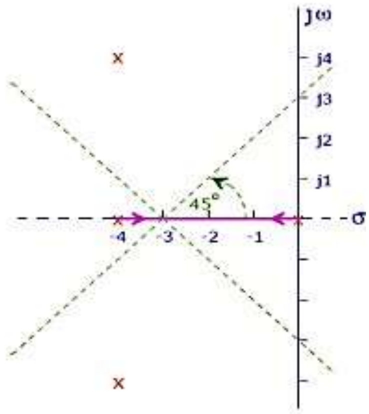
- $K = 0$ points: at $s = 0$, $s = -4$, and $S= -4+j4$ and $S= -4-j4$.

$$\left.\begin{matrix} s = 0 \\ s = -4 \\ s = -4 - j4 \\ s = -4 + j4 \end{matrix}\right\} n = 4$$

Step#2

- $K = \infty$ points: at $s \to \infty$, $s \to \infty$, $s \to \infty$ and $s \to \infty$.

$$\left.\begin{matrix} s \to \infty \\ s \to \infty \\ s \to \infty \\ s \to \infty \end{matrix}\right\} m = 0$$



Step#3

Number of asymptotes = $|n - m| = 4 - 0 = 4$

Step#4

Angle of asymptotes

$\theta_0 = 45$, $\theta_1 = 135$, $\theta_2 = 225$ and $\theta_3 = 315$

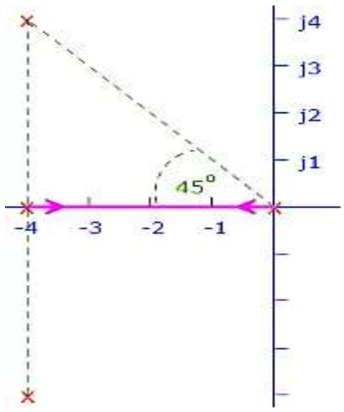
Step#5

Intersection of asymptotes with real axis:

$$\sigma = \frac{(0 - 4 - 4 - 4) - 0}{4 - 0} = -3$$

Step#6

Root locus on real axis is as shown in figure below.



Step#7

Intersection of root locus with imaginary axis:
From Routh based on the system characteristic equation:

$$S^4 + 12S^3 + 64S^2 + 128S + K = 0$$

S^4	1	64	K
S^3	12	128	
S^2	53.33	K	
S	$\frac{6826.667-12K}{53.33}$		
S^0	K		

From the first column of the above Routh array;

$K > 0$
$6826.667 - 12K > 0$
$K < 568.889$
$0 < K < 568.889$
Max. value of K for satiability K=568.889 (Critically stable system)
Auxiliary equation
$53.33\,S^2 + 568.889 = 0 \rightarrow S = \pm j\,3.266$ (frequency of sustained oscillation)

Step#8

Break away points:

$$G1(S)H1(S) = \frac{1}{S^4 + 12S^3 + 64S^2 + 128S}$$

$$\frac{dG1(S)H1(S)}{dS} = \frac{-(4S^3 + 36S^2 + 128S + 128)}{(S^4 + 12S^3 + 64S^2 + 128S)^2} = 0$$

$$4S^3 + 36S^2 + 128S + 128 = 0$$

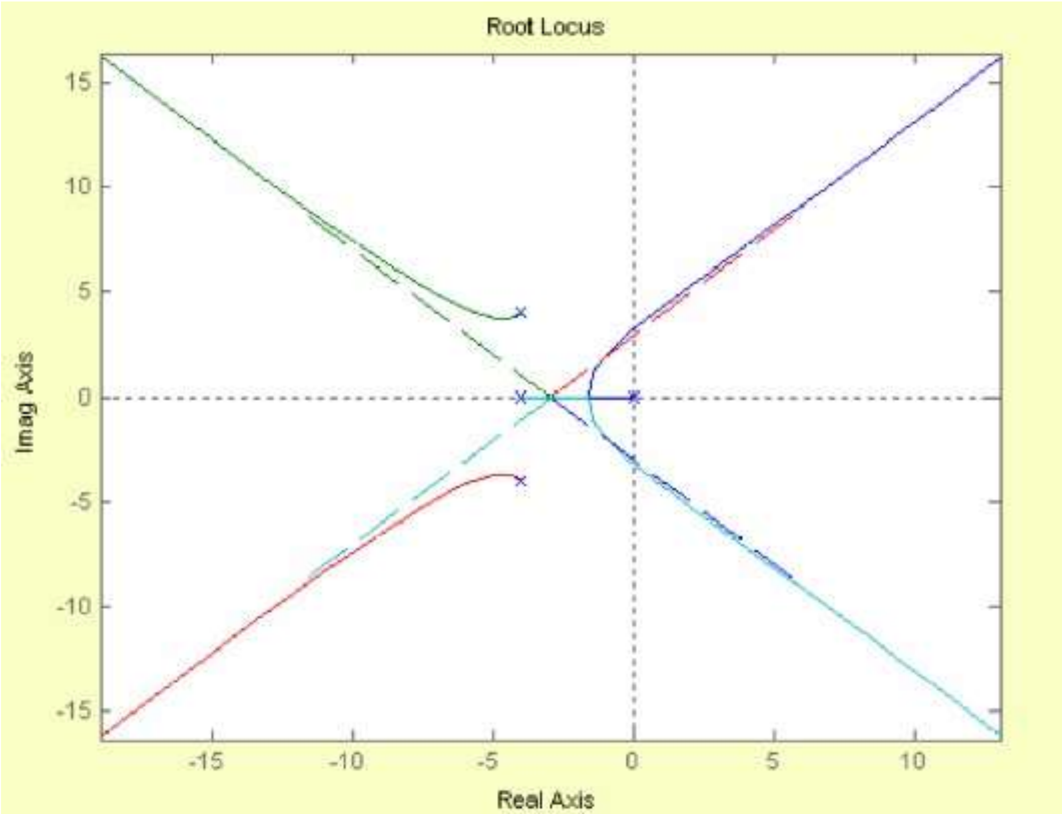
Solving the above equation for beark away points, we get:
$S1 = -1.5767$ (Accepted)
$S2 = -3.712 + j2.553$ (Rejected)
$S3 = -3.712 - j2.553$ (Rejected)

Step#9

Angle of departure for complex poles:

$$-(\theta_d + 90^\circ + 90^\circ + 135^\circ) = -180^\circ$$

$$\theta_d = -135^\circ$$



The value of K for critically damped system is at break away point

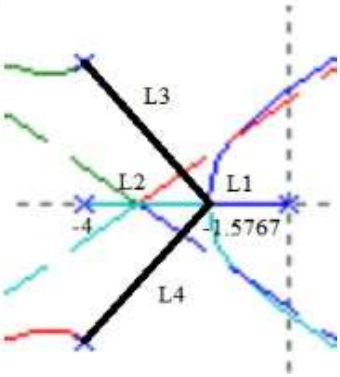
$$K = \frac{\text{حاصل ضرب أطوال الأقطاب}}{\text{حاصل ضرب أطوال الأصفار}} = \frac{L1 \times L2 \times L3 \times L4}{1}$$

L1 = 1.5767

L2 = 4-1.5767 = 2.4233

L3=L4 = $\sqrt{4^2 + 2.4233^2}$ = 4.6768

$$K = \frac{1.5767 \times 2.4233 \times 4.6768 \times 4.6768}{1} = 83.57$$



Exercise 1 :

Determine the roots locus of a control system with the following open-loop transfer function:

$G_{(s)}H_{(s)} = K / s(s + 3)(s + 9)$

Then determine the operating point, s_o, for a damping ratio of 0.5 ($\zeta = 0.5$). Also find the gain, K, and the resonant frequency, ω_o, at the operating point.