chapter one

1.1 Fourier series

Fourier series: is trigonometric function series of the form

$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ **Fourier series Formula:**

where the Fourier coefficients a_0, a_n and b_n are called constant the Euler coefficients

to find coefficients: -

1) a_0=we integrate equation

$$\int_{-\pi}^{\pi} f(x)\ dx = \int_{-\pi}^{\pi} ao\ dx + \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right)\ dx$$

$$ao = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx$$

2) an =multiply equation (1) by cos mx

$$\int_{-\pi}^{\pi} f(x) \cos mx\, dx = \int_{-\pi}^{\pi} ao\ \cos mx\, dx + \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \cos mx\right)\ dx$$

$$an = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\ dx$$

3) bn= multiply equation (1) by sin mx

$$\int_{-\pi}^{\pi} f(x) \sin mx\, dx = \int_{-\pi}^{\pi} ao\ \sin mx\, dx + \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \sin mx\right)\ dx$$

$$bn = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\ dx$$

Example 1: Find the Fourier series representation peroidc function $f(x) = x \quad and -\pi \le x \le \pi$

Solution//

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

1) $ao = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\ dx = \frac{1}{2\pi}(\frac{x^2}{2})|_{-\pi}^{\pi}$

$$ao = \frac{1}{2\pi}\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) = 0$$

2) $an = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\ dx = \frac{1}{\pi}\left(\frac{1}{n} x\sin nx + \frac{1}{n^2}\cos nx\right)|_{-\pi}^{\pi} = \frac{1}{\pi}\left(\left(\frac{\pi}{n}\sin n\pi + \frac{1}{n^2}\cos n\pi\right) - \left(\frac{\pi}{n}\sin n\pi + \frac{1}{n^2}\cos n\pi\right)\right) =$

$$an = 0$$

$$3)bn = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx = bn = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\ dx$$

$$= \frac{1}{\pi}(\frac{-x}{n} x\cos nx + \frac{1}{n^2}\sin nx)|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\left(\frac{-\pi}{n}\cos n\pi + \frac{1}{n^2}\sin n\pi\right)\right.$$

$$\left. - \left(\frac{\pi}{n}\cos n(-\pi) + \frac{1}{n^2}\sin n(-\pi)\right)\right)$$

$$= \frac{1}{\pi}\left(\frac{-2\pi}{n}\cos n\pi + \frac{2}{n^2}\sin n\pi\right) = \frac{-2}{n}\cos n\pi + \frac{2}{\pi n^2}\sin n\pi$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = 0 + 0\cos nx + \sum_{n=1}^{\infty}(b_n \sin nx)$$

$$= b1 \sin x + b2 \sin 2x + \cdots = 2\sin x - \sin 2$$

1.2 Dirichlet's Theorem.

If f(x) is aperiodic function at which at any period has finite number.

Example 1: Find the Fourier series representation peroidc function

$$f(x) = \begin{cases} -k & -\pi<x<0 \\ k & 0<x<\pi \end{cases} \quad where\ k\ is\ constant$$

Solution//

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

1) $$\boldsymbol{ao = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{2\pi}[\int_{-\pi}^{0} -k\ dx + \int_{0}^{\pi} k\ dx)] =}$$

$$\boldsymbol{\frac{1}{2\pi}[(-kx)|_{-\pi}^{0} + (kx)|_{0}^{\pi}]}$$

$$\boldsymbol{ao = \frac{1}{2\pi}(-k\pi + k\pi) = 0}$$

2)

$$\boldsymbol{an = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx =}$$

$$\boldsymbol{\frac{1}{\pi}[\int_{-\pi}^{0} -k\ \cos nx\ dx + \int_{0}^{\pi} k\ \cos nx\ dx)] = \frac{1}{\pi}[(\frac{-k\sin nx}{n})|_{-\pi}^{0} +}$$

$$\boldsymbol{(\frac{k\sin nx}{n})|_{0}^{\pi}] = \frac{1}{\pi}[(0 - \frac{k\sin n\pi}{n}) + (\frac{k\sin nx}{n} - 0)] = an =}$$

$$3)bn = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

$$= \frac{1}{\pi}[\int_{-\pi}^{0} -k\ \sin nx\, dx + \int_{0}^{\pi} k\ \sin nx\, dx)]$$

$$= \frac{1}{\pi}[(\frac{k\cos nx}{n})|_{-\pi}^{0} + (\frac{-k\cos nx}{n})|_{0}^{\pi}]$$

$$= \frac{1}{\pi}[(\frac{k}{n} - \frac{k\cos n\pi}{n}) + (\frac{-k\cos n\pi}{n} - \frac{-k}{n})]$$

$$= \frac{2k}{\pi n} - \frac{2k\cos n\pi}{\pi n} = \frac{2k}{\pi n}(1 - \cos n\pi)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = 0 + 0\cos nx + \sum_{n=1}^{\infty}(b_n \sin nx)$$

$$= \frac{4k}{\pi}\sin x + 0 + \frac{4k}{3\pi}\sin 3x + \cdots$$

1.3 Even and Odd Functions "Half-Range Expansions

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval (0,L) which is half of the interval(-L,L) thus accounting for the name half range] and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely,(-L,0).

Note:-

1. An odd function is a function with the property f (− x) = − f (x). For example :

1.f(x)= x^3 . let x = -1, then -1^3 = - 1^3

2. f (x) = sin (x). let x = - π/2, then sin (-π/2) = - sin (π/2).

Let us calculate the Fourier coefficients of an odd function: a0=an =0 but bn ≠0

$$bn = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

$$f(x) = \sum_{n=1}^{\infty} (b_n \sin nx)$$

2. An even function is a function with the property f (−x) = f (x). The sine coefficients of a Fourier series will be zero for an even function, For example :

1.f(x)= $\boldsymbol{x^2}$. let x = -1, then $\mathbf{-1^2}$ = $\mathbf{1^2}$

2. f (x) = cos (x). let x = - π, then cos (-π) = cos (π).

Let us calculate the Fourier coefficients of an even function: bn =0, but ao,an ≠0

$$\boldsymbol{ao = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ dx}$$

$$\boldsymbol{an\ = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) cosnx\ dx}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx)$$

1.4 Complex Fourier series

The complex exponential of Fourier series is obtained by equivalent of the Cosine and Sine substitution the exponential into the original form of Series

$$\boldsymbol{.\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\ and\ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = f(x)$$

$$= a_0 + \sum_{n=1}^{\infty}\left(a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2i}\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - b_n i}{2} e^{inx}\right) + \sum_{n=1}^{\infty}\left(\frac{a_n + b_n i}{2} e^{-inx}\right) = a_0 + \sum_{n=-\infty}^{\infty}(c_n e^{inx})$$

If we define

$$c_o = a_{o,} \quad , \quad c_n = \frac{a_n - b_n i}{2} \quad , \quad , \quad c_{-n} = \frac{a_n + b_n i}{2}$$

The coefficients , c_n are called complex Fourier coefficients.

They are defined by the formulas

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ e^{-inx} dx$$

$$f(x) = Co + \sum_{n=-\infty}^{\infty}(c_n\ e^{inx})$$

Example 1: Using complex form, find the Fourier series of the function

$$f(x) = \begin{cases} -1 & -\pi<x<0 \\ 1 & 0<x<\pi \end{cases}$$

Solution//

1) $co = ao = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} -1\ dx + \int_{0}^{\pi} 1\ dx)\right] = \frac{1}{2\pi}\left[(-x)\Big|_{-\pi}^{0} + (x)\Big|_{0}^{\pi}\right]$$

$$ao = \frac{1}{2\pi}(-\pi + \pi) = 0$$

2) $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ e^{-inx}dx$

$$= \frac{1}{2\pi}[\int_{-\pi}^{0} -1\ e^{-inx}dx + \int_{0}^{\pi} 1\, e^{-inx}\ dx)]$$

$$= \frac{1}{2\pi}[\int_{-\pi}^{0} -1\ \frac{e^{-inx}}{-in}dx + \int_{0}^{\pi} 1\, e^{-inx}\ dx)]$$

$$= \frac{1}{2\pi}[(-1\ \frac{e^{-inx}}{-in})|_{-\pi}^{0} + (1\frac{e^{-inx}}{-in})|_{0}^{\pi}]$$

$$c_n = \frac{1}{2\pi}\left((\frac{1}{in} - \frac{e^{in\pi}}{in}) + (\frac{e^{-in\pi}}{-in} - \frac{1}{-in}\right)$$

$$= \frac{1}{2\pi}(\frac{1 - e^{in\pi} - e^{-in\pi} + 1}{in} = \frac{2}{\pi i}$$

n=1 , $e^{i\pi} = \cos\pi + i\sin\pi = -1$, $(e^{in\pi} = -1^{(n)})$=

$$f(x) = c_o + \sum_{n=-\infty}^{\infty} (c_n\ e^{inx}) = 0 + \sum_{n=-\infty}^{\infty} \left(\frac{2}{\pi i}\ e^{ix}\right)$$

H.W. Example 1: Using complex form, find the Fourier series of the function $f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ 1 & 0 \le x \le \pi \end{cases}$