Al-Mustaqbal University

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Laplace Transforms

The Laplace transform of an expression $f(t)$ is denoted by $L\{f(t)\}$ and is defined as the semi-infinite integral

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,\mathrm{d}t \qquad (1)$$

The parameter s is assumed to be positive and large enough to ensure that the integral converges. In more advanced applications s may be complex and in such cases the real part of s must be positive and large enough to ensure convergence.

In determining the transform of an expression, you will appreciate that the limits of the integral are substituted for t, so that the result will be an expression in s. Therefore

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,\mathrm{d}t = F(s)$$

Example 1

To find the Laplace transform of $f(t) = a$ (constant).

$$L\{a\} = \int_0^{\infty} ae^{-st}\mathrm{d}t = a\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = -\frac{a}{s}\left[e^{-st}\right]_0^{\infty} = -\frac{a}{s}\{0-1\} = \frac{a}{s}$$

$$\therefore\ L\{a\} = \frac{a}{s} \qquad (s > 0)$$

Example 2

To find the Laplace transform of $f(t) = e^{at}$ (a constant). As with all cases, we multiply $f(t)$ by e^{-st} and integrate between $t = 0$ and $t = \infty$.

$$\therefore\ L\{e^{at}\} = \int_0^{\infty} e^{at}e^{-st}\,\mathrm{d}t = \int_0^{\infty} e^{-(s-a)t}\,\mathrm{d}t = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} = -\frac{1}{s-a}\{0-1\}$$

$$\therefore\ L\{e^{at}\} = \frac{1}{s-a} \qquad (s > a)$$

So we already have two standard transforms

$$L\{a\} = \frac{a}{s} \quad \text{and} \quad L\{e^{at}\} = \frac{1}{s-a}$$

And in same way we have $L\{4\} = \frac{4}{s}$; $L\{-5\} = -\frac{5}{s}$;

$$L\{e^{4t}\} = \frac{1}{s-4}; \text{ **And** } L\{e^{-2t}\} = \frac{1}{s+2}$$

Note that, as we said earlier, the Laplace transform is always an expression in s.

Example 3

To find the Laplace transform of $f(t) = \sin at$. We could, of course, apply the definition and evaluate

$$L\{\sin at\} = \int_0^\infty \sin at \cdot e^{-st}\, dt$$

using integration by parts.

However, it is much shorter if we use the fact that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

so that $\sin\theta$ is the imaginary part of $e^{j\theta}$, written $\mathscr{I}(e^{j\theta})$.

The function $\sin at$ can therefore be written $\mathscr{I}(e^{jat})$ so that

$$L\{\sin at\} = L\{\mathscr{I}(e^{jat})\} = \mathscr{I}\int_0^\infty e^{jat}e^{-st}\, dt = \mathscr{I}\int_0^\infty e^{-(s-ja)t}\, dt$$

$$= \mathscr{I}\left\{\left[\frac{e^{-(s-ja)t}}{-(s-ja)}\right]_0^\infty\right\} = \mathscr{I}\left\{-\frac{1}{(s-ja)}[0-1]\right\} = \mathscr{I}\left\{\frac{1}{s-ja}\right\}$$

We can rationalise the denominator by multiplying top and bottom by $s + ja$

$$\therefore\ L\{\sin at\} = \mathscr{I}\left\{\frac{s+ja}{s^2+a^2}\right\} = \frac{a}{s^2+a^2}$$

$$\therefore\ L\{\sin at\} = \frac{a}{s^2+a^2}$$

We can use the same method to determine $L\{\cos at\}$ since $\cos at$ is the real part of e^{jat}, written $\Re(e^{jat})$.

Then $$L\{\cos at\} = \Re\left\{\frac{s+ja}{s^2+a^2}\right\} = \frac{s}{s^2+a^2}$$

And so $$L\{\sin 2t\} = \frac{2}{s^2+4}; \qquad L\{\cos 4t\} = \frac{s}{s^2+16}$$

Example :

Laplace transforms of $f(t) = \sinh at$ and $f(t) = \cosh at$.

Starting from the exponential definitions of $\sinh at$ and $\cosh at$, i.e.

$$\sinh at = \tfrac{1}{2}(e^{at} - e^{-at}) \quad \text{and} \quad \cosh at = \tfrac{1}{2}(e^{at} + e^{-at})$$

we proceed as follows.

(a) $f(t) = \sinh at$. $\quad L\{\sinh at\} = \int_0^\infty \sinh at \, e^{-st} \, dt$

$$= \frac{1}{2}\int_0^\infty (e^{at} - e^{-at})e^{-st} \, dt = \frac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} - e^{-(s+a)t}\right\} dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2 - a^2}$$

$$\therefore \; L\{\sinh at\} = \frac{a}{s^2 - a^2}$$

(b) $f(t) = \cosh at$. Proceeding in the same way

$$L\{\cosh at\} = \frac{1}{2}\int_0^\infty (e^{at} + e^{-at})e^{-st} \, dt = \frac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} + e^{-(s+a)t}\right\} dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$

$$\therefore \; L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

Function	Laplace Transform
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{a^2+s^2}$
$\cos(at)$	$\frac{s}{a^2+s^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$f^{(2)}(t)$	$s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^n\mathcal{L}\{f(t)\} - \sum_{r=0}^{n-1} s^{n-1-r} f^{(r)}(0)$