College of Engineering
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Third Stage

# Engineering Analysis

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# *Definition of differential equations*

- A differential equation is an equation that contains one or more derivatives, such as

- $\frac{d^2y}{dx^2} + \frac{dy}{dx} += \sin x$ , $y'' + y' - \ln x = 0$ and $\dot{x} - 2\dot{} \dot{x} = 5$

# Classification of differential equations



## 1- By type:

* *Ordinary differential equation* (ODE): in which all derivatives are with respect to a single independent variable, such as

$$\frac{dy}{dx} + \sin x = x, \qquad dx + y dy = 0 \qquad and \qquad \frac{dy}{dx} + \frac{dz}{dx} = x$$

* *Partial differential equation* (PDE): in which at least one derivative is with respect to two or more independent variables, such as

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y} = x, \qquad and \qquad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

## 2- By order:

The order of the differential equation is the order of the highest derivative appears in that equation, for example

$\left(\frac{dy}{dx}\right)^3 + \sin x = x$ is a first-order ordinary differential equation (1st order ODE).

$\frac{\partial^3 y}{\partial x^3} + \frac{\partial y}{\partial x} = x$ is a third – order partial differential equation (3rd order PDE).

## 3- By degree:

The degree of the differential equation is the power of the highest derivative appears in that equation, for example

$\left(\frac{dy}{dx}\right)^2 + sin\,x = x$ is a 2nd degree, 1st order ODE.

$\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin x$ is a 1st degree, 2nd order ODE.

$\frac{d^3y}{dx^3} + \frac{dy}{dx} += \sin x$ is a 1st degree, 3rd order ODE.

# 4- By linearity:

A differential equation is said to be "linear DE" if and only if each term of the equation which contains a dependent variable and\or its derivative is of linear form. In another words a differential equation is said to be "linear DE" if:

1- The dependent variable and all its derivatives appear in a linear form.

2- There is no production of a dependent variable with one of its derivatives, or one of its derivatives with another derivative.

For example

$y''' + 5y' + y = 0$ is a linear 1st degree, 3rd order ODE

$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial x} = x$ is a non-linear 1st degree, 1st order ODE.

$\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin x$ is a non-linear 1st degree, 2nd order ODE

$\frac{\partial^3 y}{\partial x^3} + \frac{\partial y}{\partial x} = x$ is a non-linear 1st degree, 3rd order PDE.

## Solution of differential equations

1- General Solution: that contains an arbitrary constant of Integration.

2- Particular Solution: when the value of the constant of integration is determined by applying the boundary or initial conditions.

# 1- Separable variables differential equations

A first-order differential equation is said to be separable if, after solving it for the derivative,

$$\frac{dy}{dx} = F(x, y) \dots\dots(1)$$

the right-hand side can then be factored as "a formula of just x " times "a formula of just y ",

just y ",

$$F(x, y) = f(x)g(y) \dots\dots(2)$$

If this factoring is not possible, the equation is not separable.

More concisely, a first-order differential equation is separable if and only if it can be written

As
$$\frac{dy}{dx} = f(x)g(y) \dots\dots(3)$$

where f and g are known functions.

Then the solution may be found by the technique of separable of variables:

This result is obtained by dividing the standard form (eq 3) by $g(y)$ ,and then integrating both sides.

# Examples:

Find the general solution for each of the following. Where possible, write your answer as an explicit solution.

**1.** $\frac{dy}{dx} = xy - 4x$

Solution

$$\frac{dy}{dx} = xy - 4x \Leftrightarrow \frac{dy}{dx} = x(y-4) \Leftrightarrow \frac{1}{y-4}dy - xdx = 0, \text{assume} y \neq 4, \text{we obtain}$$

$$\int \frac{1}{y-4}\mathrm{dy} - \int \mathrm{xdx} \Leftrightarrow \ln|y-4| - \frac{1}{2}x^2 = C_1 \Leftrightarrow \ln|y-4| = \frac{1}{2}x^2 + C_1$$

$$|y-4| = e^{\frac{1}{2}x^2 + C_1}, let C = e^{C_1} \Leftrightarrow |y-4| = Ce^{\frac{1}{2}x^2}$$

as the general solution for the differential equation $\frac{dy}{dx} = xy - 4x$

Note:

Since $|y-4| = Ce^{\frac{1}{2}x^2} \Leftrightarrow y - 4 = \begin{cases} ce^{\frac{1}{2}x^2} & \text{if } y \geq 4 \\ -Ce^{\frac{1}{2}x^2} = Ke^{\frac{1}{2}x^2} & \text{if } y < 4 \end{cases} \Leftrightarrow y = \begin{cases} Ce^{\frac{1}{2}x^2} + 4 \\ Ke^{\frac{1}{2}x^2} + 4 \end{cases}$

**2. $\frac{dy}{dx} = 3y^2 - y^2 \sin(x)$**

Solution :

$$\text{since } \frac{dy}{dx} = y^2(3 - \sin(x)) \Leftrightarrow \frac{1}{y^2}dy - (3 - \sin(x))dx = 0, \text{we obtain that}$$

$$\int \frac{1}{y^2}dy - \int (3 - \sin(x))dx$$

$$\Leftrightarrow -\frac{1}{y} - (3x + \cos(x)) = C_1$$

$$\Leftrightarrow -\frac{1}{y} = 3x + \cos(x) + C_1$$

$$\Leftrightarrow y = -\frac{1}{3x + \cos(x) + C_1}, \text{let } C = -C_1$$

$$\Leftrightarrow y = \frac{1}{C - 3x - \cos(x)}$$

As general solution for the differential equation $\frac{dy}{dx} = 3y^2 - y^2\sin(x)$

**3.** $\frac{dy}{dx} = xy - 3x - 2y + 6$

Solution :

$$\frac{dy}{dx} = xy - 3x - 2y + 6$$

$$\Leftrightarrow \frac{dy}{dx} = (x-2)(y-3)$$

$$\Leftrightarrow \frac{1}{y-3}dy - (x-2)dx = 0$$

$$\int \frac{1}{y-3}dy - \int (x-2)dx$$

$$\Leftrightarrow \ln|y-3| - \frac{1}{2}x^2 + 2x = C_1$$

$$\Leftrightarrow \ln|y-3| = \frac{1}{2}x^2 - 2x + C_1$$

$$\Leftrightarrow |y-3| = e^{\frac{1}{2}x^2 - 2x + C_1}, \text{let } C = e^{C_1}$$

$$\Leftrightarrow |y-3| = Ce^{\frac{1}{2}x^2 - 2x}$$

Note: $|y-3| = Ce^{\frac{1}{2}x^2-2x}$

$$y - 3 = \begin{cases} ce^{\frac{1}{2}x^2-2x} \text{ if } y \geq 3 \\ -Ce^{\frac{1}{2}x^2-2x} = Ke^{\frac{1}{2}x^2-2x} \text{ if } y < 3 \end{cases}$$

$$\Leftrightarrow y = \begin{cases} Ce^{\frac{1}{2}x^2-2x} + 3 \\ Ke^{\frac{1}{2}x^2-2x} + 3 \end{cases}$$

As general solution for the differential equation $\frac{dy}{dx} = 3y^2 - y^2\sin(x)$

## 2- Homogeneous differential equations (reducible to separable DE)

A first order differential equation

$$\frac{dy}{dx} = F(x, y)$$

is called homogeneous equation, if the right side satisfies the condition

$$f(tx, ty) = f(x, y)$$

for all t. In other words, the right side is a homogeneous function (with respect to the variables x and y) of the zero order:

$$f(tx, ty) = t^0 f(x, y) = f(x, y)$$

A homogeneous differential equation can be also written in the form

$$\acute{y} = f\left(\frac{x}{y}\right)$$

## 2- Homogeneous differential equations (reducible to separable DE)

or alternatively, in the differential form:

$$P(x,y)dx + Q(x,y)dy = 0$$

where $P(x,y)$ and $Q(x,y)$ are homogeneous functions of the same degree.

The first substitution we'll take a look at will require the differential equation to be in the form,

$$\acute{y} = f\left(\frac{x}{y}\right)$$

A function f (x, y) is said to be homogeneous of degree n if;

$$f(tx,ty) = t^n f(x,y)$$

## 2- Homogeneous differential equations (reducible to separable DE)

* If $f(x,y)=2y^4-x^2y^2$, then

$$f(tx,ty)=2(ty)^4-(tx)^2(ty)^2=t^4(2y^4-x^2y^2)=t^4 f(x,y),$$

$\therefore$ $f(x,y)$ is homogeneous of degree 4.

* If $f(x,y)=\dfrac{y}{x}-3e^{x/y}+\sin\dfrac{x}{y}$, then

$$f(tx,ty)=\frac{ty}{tx}-3e^{tx/ty}+\sin\frac{tx}{ty}=t^0\left(\frac{y}{x}-3e^{x/y}+\sin\frac{x}{y}\right)=t^0 f(x,y),$$

$\therefore$ $f(x,y)$ is homogeneous of degree 0.

# 2- Homogeneous differential equations (reducible to separable DE)

The differential equation $P(x, y)dx + Q(x, y)dy = 0$ is called homogeneous if $P(x, y)$ and $Q(x, y)$ are homogeneous and of the same degree (i.e. all terms of the DE are of the same total degree in the variables $x$ $and$ $y$).

For example,

- * $x(y + x)dx = y^2 dy$, is homogeneous of degree 2.
- * $(xy + x)dx = (y^2 - x^2)dy$, is non-homogeneous.
- * $y(\ln y - \ln x - 1)dx + xdy = 0$, is homogeneous of degree 1.
- * $y(\ln y - 1)dx + xdy = 0$, is non-homogeneous.

The homogeneous DE can be always reduced to separable variables DE by the substitution

$$y = vx \text{ or } x = vy$$

# Solving Homogeneous Differential Equations

The first substitution we'll take a look at will require the differential equation to be in the form,

$$\acute{y} = f\left(\frac{x}{y}\right)$$

First order differential equations that can be written in this form are called homogeneous differential equations. Note that we will usually have to do some rewriting in order to put the differential equation into the proper form.

Once we have verified that the differential equation is a homogeneous differential equation and we've gotten it written in the proper form we will use the following substitution.

$$v(x) = \frac{y}{x}$$

We can then rewrite this as,

$$y = xv$$

# Solving Homogeneous Differential Equations



Differentiating equation

$$\acute{y} = \acute{v}x + v$$

Under this substitution the differential equation is then,

$$v + x\acute{v} = F(v) \Rightarrow$$

$$x\acute{v} = F(v) - v \quad \Rightarrow \quad \frac{dv}{F(v)-v} = \frac{dx}{x}$$

As we can see with a small rewrite of the new differential equation we will have a separable differential equation after the substitution.

# Examples:

Example 1: Show that the differential equation $\boldsymbol{x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x}$ is homogeneous and solve it ?

Solution : The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y\cos(\frac{y}{x}) + x}{x\cos(\frac{y}{x})} \quad \dots\dots\dots (1)$$

It is a differential equation of the form

$$F(x, y) = \frac{dy}{dx}$$

Here $\qquad F(x, y) = \frac{y\cos(\frac{y}{x})+x}{x\cos(\frac{y}{x})}$

Replacing x by tx and y by ty, we get $\qquad F(tx, ty) = \frac{t[y\cos(\frac{y}{x})+x]}{t(x\cos\frac{y}{x})} = t^0[F(x, y)]$

Thus, F (x, y) is a homogeneous function of degree zero.

Therefore, the given differential equation is a homogeneous differential equation.

To solve it we make the substitution

$y = xv \qquad \dots\dots\dots(2)$

# Examples:

Differentiating equation (2) with respect to, $x$ we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \dots\dots\dots (3)$$

Substituting the value of $y$ and $\frac{dy}{dx}$ in equation (1), we get

$$v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v$$

$$x\frac{dv}{dx} = \frac{1}{\cos v} \Rightarrow \cos v dv = \frac{dx}{x} \Rightarrow \int \cos v \, dv = \int \frac{1}{x} dx$$

$$\Rightarrow \quad \sin v = ln|x| + c$$

Replacing v by $\frac{\mathrm{y}}{\mathrm{x}}$, we get $\quad \sin\left(\frac{\mathrm{y}}{\mathrm{x}}\right) = \ln|\mathrm{x}| + \mathrm{c}$

# Examples:

Example 2: Solve $\mathbf{2(2x^2 + y^2)dx - xydy = 0}$

Solution

The given DE is homogeneous of degree 2.

Let $y = xv \Rightarrow \quad dy = vdx + xdv$

$$2(2x^2 + (xv)^2)dx - x(xv)(vdx + xdv) = 0$$

$$\Rightarrow (4x^2 + 2v^2x^2)dx - v^2x^2dx - vx^3dv = 0$$

$$\Rightarrow (4x^2 + v^2x^2)dx - vx^3dv = 0$$

$$\Rightarrow (4 + v^2)dx - vxdv = 0$$

$$\frac{dx}{x} - \frac{v}{4+v^2} = 0 \Rightarrow \ln x - \frac{1}{2}\ln(4+v^2) = C_1$$

$$2\ln x - \ln(4+v^2) = 2C_1 \Rightarrow \ln\frac{x^2}{4+v^2} = 2C_1 \Rightarrow \frac{x^2}{4+v^2} = e^{2C_1}$$

$$\frac{x^2}{4+v^2} = C \quad [C = e^{2C_1}] \Rightarrow x^2 = C(4+v^2) \Rightarrow x^2 = C\left(4 + \left(\frac{y}{x}\right)^2\right)$$

$$\therefore x^4 = C(4x^2 + y^2)$$

Note: * The given DE can also be solved by letting $x = vy$.

# Reductible to homogeneous DE



Consider the DE $(a_1x+b_1y+c_1)dx \pm (a_2x+b_2y+c_2)dy=0.$

If $c_1=c_2=0$, then the given DE is homogeneous.

If $c_1\neq 0$ or $c_2\neq 0$, the given DE is nonhomogeneous, then consider the lines:

$$a_1x+b_1y+c_1=0 \quad \text{and} \quad a_2x+b_2y+c_2=0.$$

* If $\left(\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\right)$, then the two lines intersect at a point such as $p(h,k)$, and the given DE can be reduced to a homogeneous DE by the two substitutions:

$$x=x^*+h \quad \text{and} \quad y=y^*+k.$$

* If $\left(\frac{a_1}{a_2}=\frac{b_1}{b_2}=r\right)$, then the two lines are parallel, and the given DE becomes

$$[r(a_2x+b_2y)+c_1]dx \pm [(a_2x+b_2y)+c_2]dy=0,$$

and the given DE can be reduced to a separable variables DE by the substitution:

$$z=a_2x+b_2y.$$

**Examples:**

1: Solve $(\boldsymbol{x-4y-3})\boldsymbol{dx} - (\boldsymbol{x-6y-5})\boldsymbol{dy} = \boldsymbol{0}$.

Solution

$$\frac{a_1}{a_2} = \frac{1}{1} = 1 \qquad \text{and} \qquad \frac{b_1}{b_2} = \frac{-4}{-6} = \frac{2}{3}$$

Since $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ , then the two lines intersect. To find the point of intersection,

$\boldsymbol{x-4y-3}$ ..................(1)

$\boldsymbol{x-6y-5}$ ..................(2)

Subtracting Eq.2 from Eq.1 gives $\boldsymbol{2y+2=0} \Longrightarrow \boldsymbol{y=-1} \Longrightarrow \boldsymbol{x=-1}$

Thus the point of intersection $p(h,k) is\ (p(-1,-1)$

$$\therefore Let\ x = x^* + h \Longrightarrow \boldsymbol{dx} = dx^*$$

$$and\ y = y^* + k \Longrightarrow \boldsymbol{dy} = dy^*$$

$$\therefore \quad [(x^*-1)-4(y^*-1)-3]dx^* - [(x^*-1)-6(y^*-1)-5]dy^* = 0,$$

$$\Rightarrow \quad (x^*-4y^*)dx^* - (x^*-6y^*)dy^* = 0. \qquad \text{(Homogeneous DE)}$$

2: Solve $(2x+3y+4)dx - (4x+6y+1)dy = 0$.

Solution : $\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2}$ and $\frac{b_1}{b_2} = \frac{3}{6} = \frac{1}{2}$

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ , then the two lines are parallel.

$$(2x+3y+4)dx - (4x+6y+1)dy = 0,$$

Let $z = 2x+3y \Rightarrow dz = 2dx + 3dy \Rightarrow dy = \frac{1}{3}(dz - 2dx)$

$$\therefore (z+4)dx - (2z+1)\left(\frac{1}{3}(dz-2dx)\right) = 0,$$

$$\Rightarrow 7(z+2)dx - (2z+1)dz = 0, \qquad \text{(Separable DE.)}$$

$$\Rightarrow 7dx - \frac{2z+1}{z+2}dz = 0 \Rightarrow \int 7dx - \int \frac{2z+1}{z+2} = 0.$$

For $\frac{2z+1}{z+2} = \frac{2(z+2)-3}{z+2} = 2 - \frac{3}{z+2}$

$$\therefore \int 7dx - \int 2 - \frac{3}{z+2} = 0 \Rightarrow 7x - 2z + 3\ln(z+2) = C_1$$

$$7x - 2(2x+3y) + 3\ln(2x+3y+2) = C_1$$

$$\Rightarrow \ln(2x+3y+2) = 2y - x + \frac{C_1}{3}$$

$$\Rightarrow \ln(2x+3y+2) = 2y - x + C. \qquad \left[C = \frac{C_1}{3}\right]$$