

COLLEGE OF ENGINEERING
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Engineering Analysis

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3- EXACT DIFFERENTIAL EQUATIONS

Theorem:

The differential equation $M(x, y)dx + N(x, y)dy = 0$ is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof:

Let $f(x, y) = C$ be any function, then the total differentiation (exact differential) of $f(x, y)$ is given by;

$$df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy = 0 \Rightarrow$$

Let $\frac{\partial f}{\partial x} = M(x, y)$ and $\frac{\partial f}{\partial y} = N(x, y)$, then \Rightarrow

$$M(x, y)dx + N(x, y)dy = 0$$



We have $\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} \Rightarrow$ But $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \Rightarrow \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

❖ **To solve an exact DE we use the following procedure;**

$$\frac{\partial f}{\partial x} = M(x, y) \Rightarrow f(x, y) = \int M(x, y) \cdot dx + g(y)$$

since $f(x, y) = C \Rightarrow \int M(x, y) \cdot dx + g(y) = C$

To find $g(y)$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[\int M(x, y) \cdot dx \right] + g'(y), \text{ but } \frac{\partial f}{\partial y} = N(x, y)$$

$$\therefore \frac{\partial}{\partial y} \left[\int M(x, y) \cdot dx \right] + g'(y) = N(x, y) \Rightarrow g'(y) = N(x, y) - \frac{\partial}{\partial y} \left[\int M(x, y) \cdot dx \right]$$

$$\therefore g(y) = \int \left[N(x, y) - \frac{\partial}{\partial y} \left[\int M(x, y) \cdot dx \right] \right] dy$$



Example 1: SOLVE $(3x^2y + 2xy)dx + (x^3 + x^2 + 2y)dy = 0$.

Solution:

$$M(x, y) = 3x^2y + 2xy \Rightarrow \frac{\partial M}{\partial y} = 3x^2 + 2x ; N(x, y) = x^3 + x^2 + 2y \Rightarrow \frac{\partial N}{\partial x} = 3x^2 + 2x$$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the given DE is an exact DE

Solution I:

$$M(x, y) = \frac{\partial f}{\partial x} = 3x^2y + 2xy \Rightarrow f(x, y) = x^3y + x^2y + g(y)$$

$$\frac{\partial f}{\partial y} = x^3 + x^2 + g'(y), \text{ BUT } \frac{\partial f}{\partial y} = N(x, y)$$

$$\therefore x^3 + x^2 + g'(y) = N(x, y) = x^3 + x^2 + 2y \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2$$

$$\therefore f(x, y) = x^3y + x^2y + y^2, \text{ BUT } f(x, y) = C$$

$$\therefore x^3y + x^2y + y^2 = C$$



Solution II:

$$N(x, y) = \frac{\partial f}{\partial y} = x^3 + x^2 + 2y \Rightarrow f(x, y) = x^3y + x^2y + y^2 + q(x)$$

$$\frac{\partial f}{\partial x} = 3x^2y + 2xy + q'(x), \text{ but } \frac{\partial f}{\partial x} = M(x, y)$$

$$\therefore 3x^2y + 2xy + q'(x) = M(x, y) = 3x^2y + 2xy \Rightarrow q'(x) = 0 \Rightarrow q(x) = C_1$$

$$\therefore f(x, y) = x^3y + x^2y + y^2 + C_1, \text{ but } f(x, y) = C_2$$

$$\therefore x^3y + x^2y + y^2 + C_1 = C_2 \Rightarrow x^3y + x^2y + y^2 = C. [C = C_2 - C_1]$$

Solution III:

$$M(x, y) = \frac{\partial f}{\partial x} = 3x^2y + 2xy \Rightarrow f(x, y) = x^3y + x^2y + g(y)$$

$$N(x, y) = \frac{\partial f}{\partial y} = x^3 + x^2 + 2y \Rightarrow f(x, y) = x^3y + x^2y + y^2 + q(x)$$

Comparing the above two expressions of $f(x, y)$ yields,

$$g(y) = y^2 \text{ and } q(x) = 0$$

$$\therefore f(x, y) = x^3y + x^2y + y^2, \text{ but } f(x, y) = C$$

$$\therefore x^3y + x^2y + y^2 = C$$



Example 2: Solve $(x^2 \cos xy + e^y)dy + (xycos xy + \sin xy)dx = 0$

Solution :

$$M(x, y) = xycos xy + \sin xy \Rightarrow \frac{\partial M}{\partial y} = x[y(-\sin xy)x + \cos xy] + \cos xy(x) \\ = -x^2 y \sin xy + 2x \cos xy$$

$$N(x, y) = x^2 \cos xy + e^y \Rightarrow \frac{\partial N}{\partial x} = x^2(-\sin xy)y + \cos xy(2x) + 0 \\ = -x^2 y \sin xy + 2x \cos xy$$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the given DE is an exact DE .

$$N = \frac{\partial f}{\partial y} = x^2 \cos xy + e^y \Rightarrow f = x \sin xy + e^y + g(x)$$

$$\frac{\partial f}{\partial x} = x(\cos xy)y + \sin xy + 0 + g'(x), \text{ but } \frac{\partial f}{\partial x} = M$$

$$\therefore xycos xy + \sin xy + g'(x) = M = xycos xy + \sin xy \Rightarrow$$

$$g'(x) = 0 \Rightarrow g(x) = C_1$$

$$\therefore x \sin xy + e^y + C_1 = C_2 \Rightarrow x \sin xy + e^y = C. [C = C_2 - C_1]$$



REDUCIBLE TO EXACT DIFFERENTIAL EQUATIONS

- The differential equation $M(x, y)dx + N(x, y)dy = 0$ which is not exact (i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$) can be reduced to exact DE by multiplying it by a suitable function $\mu(x, y)$ which is called **integrating factor (I.F)**

$$\mu M(x, y)dx + \mu N(x, y)dy = 0$$

The above new DE is exact if $\frac{\partial}{\partial y} (\mu M) = \frac{\partial}{\partial x} (\mu N)$

$$\frac{\partial M}{\partial y} \mu + \frac{\partial \mu}{\partial y} M = \frac{\partial N}{\partial x} \mu + \frac{\partial \mu}{\partial x} N$$

The integrating factor $\mu(x, y)$ may be a function of x only, function of y only, or a function of both x and y .

There are two methods to find the integrating factor:



i- If $\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$ is a function of x only ($\mu(x, y)$ is a function of x only),

$\therefore \frac{\partial \mu}{\partial y} = 0$ and $\frac{\partial \mu}{\partial x} = \frac{d\mu}{dx}$, then Eq.(1) becomes

$$\mu \frac{\partial M}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{d\mu}{dx} \quad \Rightarrow \quad \mu \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = N \frac{d\mu}{dx},$$

$$\Rightarrow \quad \frac{d\mu}{\mu} = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx \quad \Rightarrow \quad \ln \mu = \int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx$$

$$\therefore \mu = e^{\int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx}.$$



Example 1: Solve $(x + 3y^2)dx + 2xydy = 0$.

Solution :

$$M(x, y) = x + 3y^2 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 6y,$$

$$N(x, y) = 2xy \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2y.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the given DE is not exact.

Check,
$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{2xy} (6y - 2y) = \frac{2}{x} \quad (\text{function of } x \text{ only})$$

$$\therefore \mu = e^{\int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx} \Rightarrow \mu = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2} = x^2.$$

Multiplying the given DE by the above integrating factor (I.F) gives

$$x^2(x + 3y^2)dx + x^2(2xy)dy = 0 \quad \Rightarrow \quad (x^3 + 3x^2y^2)dx + 2x^3ydy = 0.$$

Check,
$$M(x, y) = x^3 + 3x^2y^2 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 6x^2y,$$

$$N(x, y) = 2x^3y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 6x^2y.$$



Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the given DE is reduced to exact one.

$$M = \frac{\partial f}{\partial x} = x^3 + 3x^2y^2 \quad \Rightarrow \quad f = \frac{x^4}{4} + x^3y^2 + g(y),$$

$$\frac{\partial f}{\partial y} = 2x^3y + g'(y), \quad \text{but} \quad \frac{\partial f}{\partial y} = N,$$

$$\therefore 2x^3y + g'(y) = N = 2x^3y \quad \Rightarrow \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = C_1,$$

$$\therefore f = \frac{x^4}{4} + x^3y^2 + C_1, \quad \text{but} \quad f = C_2,$$

$$\therefore \frac{x^4}{4} + x^3y^2 + C_1 = C_2 \quad \Rightarrow \quad \frac{x^4}{4} + x^3y^2 = C_3, \quad [C_3 = C_2 - C_1]$$

or $x^4 + 4x^3y^2 = C. \quad [C = 4C_3]$



Example 2: Solve $(\sin y + x^2 + 2x)dx = \cos y dy$.

Solution :

$$(\sin y + x^2 + 2x)dx = \cos y dy \quad \Rightarrow \quad (\sin y + x^2 + 2x)dx - \cos y dy = 0.$$

$$M(x, y) = \sin y + x^2 + 2x \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \cos y,$$

$$N(x, y) = -\cos y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 0.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the given DE is not exact.

Check,
$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{-\cos y} (\cos y - 0) = -1 \quad (\text{function of } x \text{ only})$$

$$\therefore \mu = e^{\int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx} \quad \Rightarrow \quad \mu = e^{\int (-1) dx} = e^{-x}.$$

Multiplying the given DE by the above integrating factor (I.F) gives

$$e^{-x} (\sin y + x^2 + 2x)dx - e^{-x} \cos y dy = 0.$$



$$N = \frac{\partial f}{\partial y} = -e^{-x} \cos y \quad \Rightarrow \quad f = -e^{-x} \sin y + g(x),$$

$$\frac{\partial f}{\partial x} = e^{-x} \sin y + g'(x), \quad \text{but} \quad \frac{\partial f}{\partial x} = M,$$

$$\therefore e^{-x} \sin y + g'(x) = M = e^{-x} (\sin y + x^2 + 2x) \quad \Rightarrow \quad g'(x) = x^2 e^{-x} + 2x e^{-x},$$

$$\therefore g(x) = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} - 2x e^{-x} - 2e^{-x} \quad \Rightarrow \quad g(x) = -x^2 e^{-x} - 4x e^{-x} - 4e^{-x},$$

$$\therefore f = -e^{-x} \sin y - x^2 e^{-x} - 4x e^{-x} - 4e^{-x}, \quad \text{but} \quad f = C,$$

$$\therefore -e^{-x} \sin y - x^2 e^{-x} - 4x e^{-x} - 4e^{-x} = C_1,$$

$$\text{or} \quad x^2 + 4x + 4 + \sin y = C e^x. \quad [C = -C_1]$$

Note;

<u>u</u>	<u>dv</u>
x^2	e^{-x}
$2x$	$-e^{-x}$
2	e^{-x}
0	$-e^{-x}$
$= x^2(-e^{-x}) - 2x e^{-x} + 2(-e^{-x})$	
$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x}$	

<u>u</u>	<u>dv</u>
$2x$	e^{-x}
2	$-e^{-x}$
0	e^{-x}
$= 2x(-e^{-x}) - 2e^{-x}$	
$= -2x e^{-x} - 2e^{-x}$	

ii- If $\frac{1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$ is a function of y only ($\mu(x, y)$ is a function of y only),

$\therefore \frac{\partial \mu}{\partial x} = 0$ and $\frac{\partial \mu}{\partial y} = \frac{d\mu}{dy}$, then Eq.(1) becomes

$$\mu \frac{\partial M}{\partial y} + M \frac{d\mu}{dy} = \mu \frac{\partial N}{\partial x} \quad \Rightarrow \quad \mu \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = -M \frac{d\mu}{dy},$$

$$\Rightarrow \quad \frac{d\mu}{\mu} = \frac{-1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dy \quad \Rightarrow \quad \ln \mu = \int \frac{-1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dy$$

$$\therefore \mu = e^{-\int \frac{1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dy}.$$



Example 1: Solve $(y + 2x)dx + x(y + x + 1)dy = 0$.

Solution :

$$M(x, y) = y + 2x \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 1,$$

$$N(x, y) = xy + x^2 + x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = y + 2x + 1.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the given DE is not exact.

$$\text{Check, } \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1 - (y + 2x + 1)}{xy + x^2 + x} = \frac{-(y + 2x)}{x(y + x + 1)} \quad (\text{is not function of } x \text{ only})$$

$$\text{Check, } \frac{1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1 - (y + 2x + 1)}{y + 2x} = \frac{-(y + 2x)}{y + 2x} = -1 \quad (\text{function of } y \text{ only})$$

$$\therefore \mu = e^{-\int \frac{1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dy} \Rightarrow \mu = e^{-\int (-1) dy} = e^y.$$

Multiplying the given DE by the above integrating factor (I.F) gives

$$e^y (y + 2x)dx + xe^y (y + x + 1)dy = 0.$$

$$M = \frac{\partial f}{\partial x} = ye^y + 2xe^y \quad \Rightarrow \quad f = xye^y + x^2e^y + g(y),$$

$$\frac{\partial f}{\partial y} = xye^y + xe^y + x^2e^y + g'(y), \quad \text{but} \quad \frac{\partial f}{\partial y} = N,$$

$$\therefore xye^y + xe^y + x^2e^y + g'(y) = N = xye^y + x^2e^y + xe^y \Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1,$$

$$\therefore f = xye^y + x^2e^y + C_1, \quad \text{but} \quad f = C_2,$$

$$\therefore xye^y + x^2e^y + C_1 = C_2 \Rightarrow xy + x^2 = Ce^{-y}. \quad [C = C_2 - C_1]$$



Example 2: Solve $\cos y \cdot dx + (2x \sin y - \cos^3 y) \cdot dy = 0$.

Solution :

$$M = \cos y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = -\sin y,$$

$$N = 2x \sin y - \cos^3 y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2 \sin y.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the given DE is not exact.

$$\text{Check, } \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{-\sin y - 2 \sin y}{2x \sin y - \cos^3 y} = \frac{-3 \sin y}{2x \sin y - \cos^3 y} \text{ (is not function of } x \text{ only)}$$

$$\text{Check, } \frac{1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{-\sin y - 2 \sin y}{\cos y} = \frac{-3 \sin y}{\cos y}, \quad \text{(function of } y \text{ only)}$$

$$\therefore \mu = e^{-\int \frac{1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dy} \Rightarrow \mu = e^{-\int \frac{-3 \sin y}{\cos y} dy} = e^{-3 \ln \cos y} = \cos^{-3} y = \frac{1}{\cos^3 y}.$$



Multiplying the given DE by the above integrating factor (I.F) gives

$$\frac{1}{\cos^3 y}(\cos y).dx + \frac{1}{\cos^3 y}(2x \sin y - \cos^3 y).dy = 0 \Rightarrow \frac{1}{\cos^2 y}.dx + \left(\frac{2x \sin y}{\cos^3 y} - 1\right).dy = 0$$

$$M = \frac{\partial f}{\partial x} = \frac{1}{\cos^2 y} \quad \Rightarrow \quad f = \frac{x}{\cos^2 y} + g(y),$$

$$\frac{\partial f}{\partial y} = -2x \cos^{-3} \cdot (-\sin y) + g'(y) = \frac{2x \sin y}{\cos^3 y} + g'(y), \quad \text{but} \quad \frac{\partial f}{\partial y} = N,$$

$$\therefore \frac{2x \sin y}{\cos^3 y} + g'(y) = N = \frac{2x \sin y}{\cos^3 y} - 1 \quad \Rightarrow \quad g'(y) = -1 \quad \Rightarrow \quad g(y) = -y,$$

$$\therefore f = \frac{x}{\cos^2 y} - y, \quad \text{but} \quad f = C,$$

$$\therefore \frac{x}{\cos^2 y} - y = C, \quad \text{or} \quad x = (y + C)\cos^2 y.$$



iii- If the above two integrating factors doesn't exists, suppose that the integrating factors in the form $\mu(x, y) = x^m y^n$ and we find the valves of m and n that make the differential equation $M(x, y)dx + N(x, y)dy = 0$ which is exact.

Example :solve the differential equation $(x^2 + xy^2)y' - 3xy + 2y^3 \dots \dots \dots (1)$

Solution

$$\left. \begin{array}{l} M = -3xy + 2y^2 \Rightarrow \frac{\partial M}{\partial y} = -3x + 6y \\ N = (x^2 + xy^2) \Rightarrow \frac{\partial N}{\partial x} = 2x + y^2 \end{array} \right\} \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{2x + y^2 - (-3x + 6y)}{-3xy + 2y^2} = \frac{5(x - y^2)}{y(2y^2 - 3x)} \quad (\text{is not function of } y \text{ only})$$

$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{-3x + 6y - (2x + y^2)}{x^2 + xy^2} = \frac{5(y^2 - x)}{x(x + y^2)} \quad (\text{is not function of } x \text{ only})$$



Suppose that $\mu(x, y) = x^m y^n$ is an integrating factor for differential equation $M(x, y)dx + N(x, y)dy = 0$ we get \Rightarrow

$$2x^m y^{n+3} - 3x^{m+1} y^{n+1} + (x^{m+2} y^n + x^{m+1} y^{n+2})y' = 0 \dots (2) \quad (\text{the DE is exact})$$

$$\frac{\partial M}{\partial y} = 2(n+3)x^m y^{n+2} - 3(n+1)x^{m+1} y^n = x^m y^n [(2n+6)y^2 - (3n+3)x]$$

$$\frac{\partial N}{\partial x} = (m+2)x^{m+1} y^n + (m+1)x^m y^{n+2} = x^m y^n [(m+2)x + (m+1)y^2]$$

Since equation (2) is exact we have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$

$$(2n+6)y^2 - (3n+3)x = (m+2)x + (m+1)y^2$$

$$\left. \begin{array}{l} m+2 = -3n-3 \\ m+1 = 2n+6 \end{array} \right\} \Rightarrow m = 1, \quad n = -2$$

The integrating factor is $\mu(x, y) = xy^{-2}$

$$(2xy - 3x^2 y^{-1}) + (x^3 y^{-2} + x^2)y' = 0$$



$$(2xy - 3x^2y^{-1}) + (x^3y^{-2} + x^2)y' = 0$$

$$\left. \begin{aligned} M = 2xy - 3x^2y^{-1} &\Rightarrow \frac{\partial M}{\partial y} = 2x + 3x^2y^{-2} \\ N = x^3y^{-2} + x^2 &\Rightarrow \frac{\partial N}{\partial x} = 3x^2y^{-2} + 2x \end{aligned} \right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The DE is exact

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