Ministry of Higher Education
Al Mustaqbal College
Biomedical Engineering Department

Lecture no. 6

Subject	**Control System 2**
Stage	**4th Stage**

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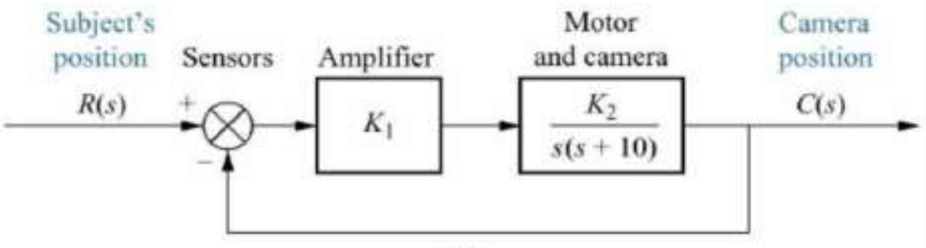
Lecture no. 6 : Root-Locus Plots

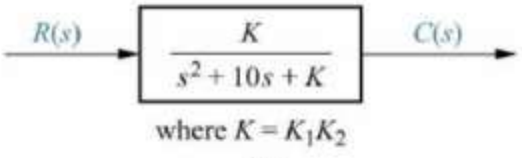
Introduction

The basic characteristic of the transient response of a closed-loop system is closely related to the location of the closed-loop poles. If the system has a variable loop gain, then the location of the closed-loop poles depends on the value of the loop gain chosen. It is important, therefore, that the designer know how the closed-loop poles move in the s plane as the loop gain is varied. From the design viewpoint, in some systems simple gain adjustment may move the closed-loop poles to desired locations.Then the design problem may become the selection of an appropriate gain value. If the gain adjustment alone does not yield a desired result, addition of a compensator to the system will become necessary

For this system



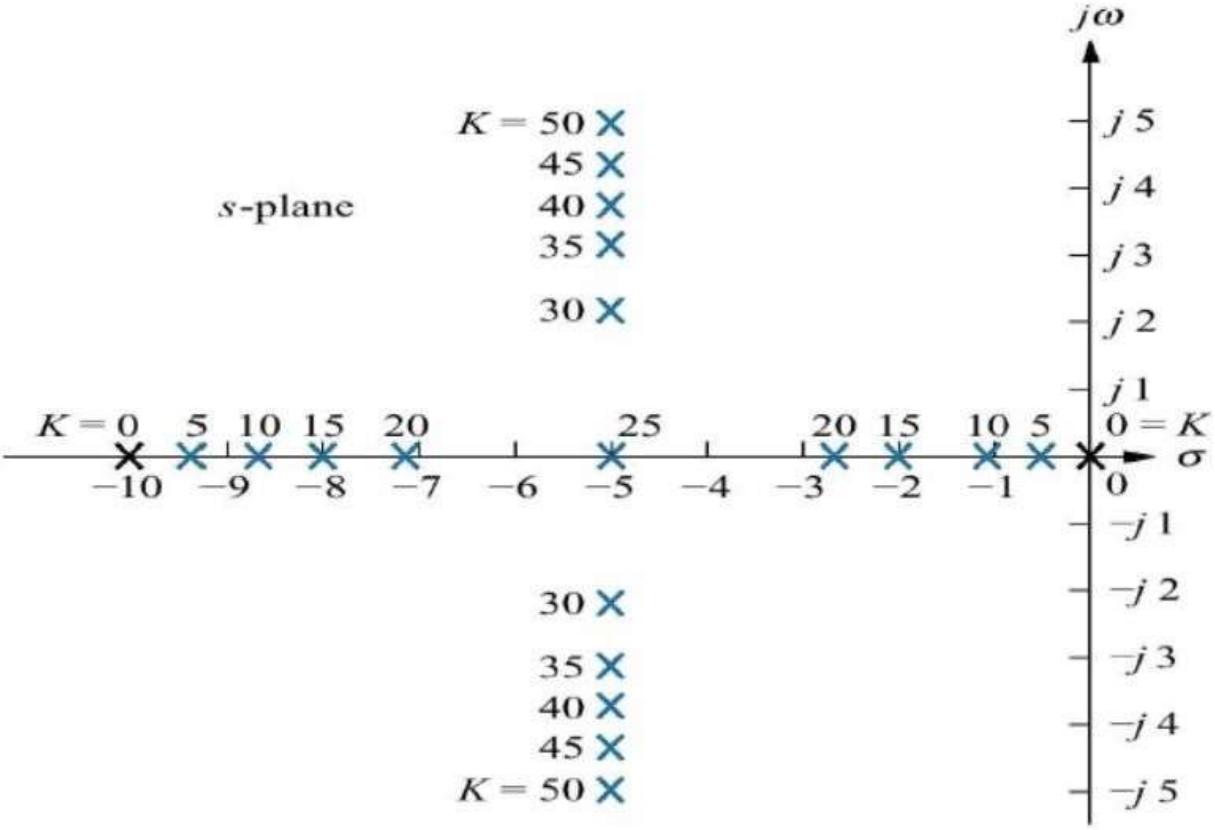
(b)



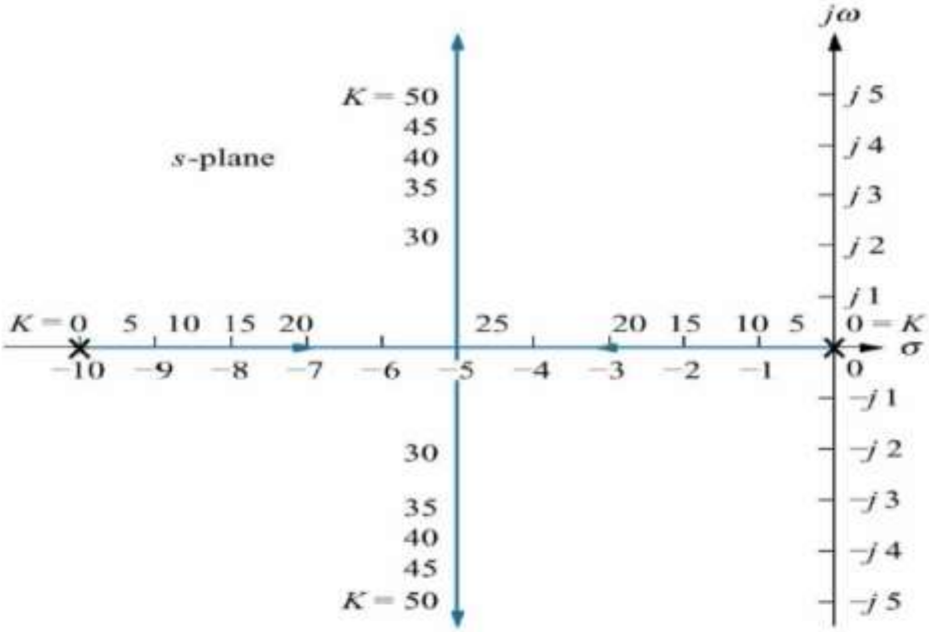
(c)

The location of poles as a function of K can be calculated as

K	**Pole 1**	**Pole 2**
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$



(*a*)



(*b*)

General Rules for Constructing Root Loci. For a complicated system with many open-loop poles and zeros, constructing a root-locus plot may seem complicated, but actually it is not difficult if the rules for constructing the root loci are applied. By locating particular points and asymptotes and by computing angles of departure from complex poles and angles of arrival at complex zeros, we can construct the general form of the root loci without difficulty.

We shall now summarize the general rules and procedure for constructing the root loci of the negative feedback control system shown in Figure 6–11.

First, obtain the characteristic equation

$$1 + G(s)H(s) = 0$$

Then rearrange this equation so that the parameter of interest appears as the multiplying factor in the form

$$1 + \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} = 0 \tag{6–11}$$

In the present discussions, we assume that the parameter of interest is the gain K, where $K > 0$. (If $K < 0$, which corresponds to the positive-feedback case, the angle condition must be modified. See Section 6–4.) Note, however, that the method is still applicable to systems with parameters of interest other than gain. (See Section 6–6.)

1. *Locate the poles and zeros of $G(s)H(s)$ on the s plane. The root-locus branches start from open-loop poles and terminate at zeros (finite zeros or zeros at infinity).* From the factored form of the open-loop transfer function, locate the open-loop poles and zeros in the s plane. [Note that the open-loop zeros are the zeros of $G(s)H(s)$, while the closed-loop zeros consist of the zeros of $G(s)$ and the poles of $H(s)$.]

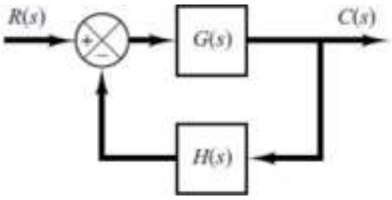


Figure 6–11
Control system.

Note that the root loci are symmetrical about the real axis of the s plane, because the complex poles and complex zeros occur only in conjugate pairs.

A root-locus plot will have just as many branches as there are roots of the characteristic equation. Since the number of open-loop poles generally exceeds that of zeros, the number of branches equals that of poles. If the number of closed-loop poles is the same as the number of open-loop poles, then the number of individual root-locus branches terminating at finite open-loop zeros is equal to the number m of the open-loop zeros. The remaining $n-m$ branches terminate at infinity ($n-m$ implicit zeros at infinity) along asymptotes.

If we include poles and zeros at infinity, the number of open-loop poles is equal to that of open-loop zeros. Hence we can always state that the root loci start at the poles of $G(s)H(s)$ and end at the zeros of $G(s)H(s)$, as K increases from zero to infinity, where the poles and zeros include both those in the finite s plane and those at infinity.

2. *Determine the root loci on the real axis.* Root loci on the real axis are determined by open-loop poles and zeros lying on it. The complex-conjugate poles and complex-conjugate zeros of the open-loop transfer function have no effect on the location of the root loci on the real axis because the angle contribution of a pair of complex-conjugate poles or complex-conjugate zeros is 360° on the real axis. Each portion of the root locus on the real axis extends over a range from a pole or zero to another pole or zero. In constructing the root loci on the real axis, choose a test point on it. If the total number of real poles and real zeros to the right of this test point is odd, then this point lies on a root locus. If the open-loop poles and open-loop zeros are simple poles and simple zeros, then the root locus and its complement form alternate segments along the real axis.

3. *Determine the asymptotes of root loci.* If the test point s is located far from the origin, then the angle of each complex quantity may be considered the same. One open-loop zero and one open-loop pole then cancel the effects of the other. Therefore, the root loci for very large values of s must be asymptotic to straight lines whose angles (slopes) are given by

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ(2k+1)}{n-m} \qquad (k = 0, 1, 2, \ldots)$$

where n = number of finite poles of $G(s)H(s)$

m = number of finite zeros of $G(s)H(s)$

Here, $k=0$ corresponds to the asymptotes with the smallest angle with the real axis. Although k assumes an infinite number of values, as k is increased the angle repeats itself, and the number of distinct asymptotes is $n-m$.

All the asymptotes intersect at a point on the real axis. The point at which they do so is obtained as follows: If both the numerator and denominator of the open-loop transfer function are expanded, the result is

$$G(s)H(s) = \frac{K[s^m + (z_1 + z_2 + \cdots + z_m)s^{m-1} + \cdots + z_1 z_2 \cdots z_m]}{s^n + (p_1 + p_2 + \cdots + p_n)s^{n-1} + \cdots + p_1 p_2 \cdots p_n}$$

If a test point is located very far from the origin, then by dividing the denominator by the numerator, it is possible to write $G(s)H(s)$ as

$$G(s)H(s) = \frac{K}{s^{n-m} + [(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)]s^{n-m-1} + \cdots}$$

or

$$G(s)H(s) = \frac{K}{\left[s + \dfrac{(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)}{n - m}\right]^{n-m}} \qquad (6\text{–}12)$$

The abscissa of the intersection of the asymptotes and the real axis is then obtained by setting the denominator of the right-hand side of Equation (6–12) equal to zero and solving for s, or

$$s = -\frac{(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)}{n - m} \qquad (6\text{–}13)$$

[Example 6–1 shows why Equation (6–13) gives the intersection.] Once this intersection is determined, the asymptotes can be readily drawn in the complex plane.

It is important to note that the asymptotes show the behavior of the root loci for $|s| \gg 1$. A root-locus branch may lie on one side of the corresponding asymptote or may cross the corresponding asymptote from one side to the other side.

4. *Find the breakaway and break-in points.* Because of the conjugate symmetry of the root loci, the breakaway points and break-in points either lie on the real axis or occur in complex-conjugate pairs.

If a root locus lies between two adjacent open-loop poles on the real axis, then there exists at least one breakaway point between the two poles. Similarly, if the root locus lies between two adjacent zeros (one zero may be located at $-\infty$) on the real axis, then there always exists at least one break-in point between the two zeros. If the root locus lies between an open-loop pole and a zero (finite or infinite) on the real axis, then there may exist no breakaway or break-in points or there may exist both breakaway and break-in points.

Suppose that the characteristic equation is given by

$$B(s) + KA(s) = 0$$

The breakaway points and break-in points correspond to multiple roots of the characteristic equation. Hence, as discussed in Example 6–1, the breakaway and break-in points can be determined from the roots of

$$\frac{dK}{ds} = -\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)} = 0 \qquad (6\text{–}14)$$

where the prime indicates differentiation with respect to s. It is important to note that the breakaway points and break-in points must be the roots of Equation (6–14), but not all roots of Equation (6–14) are breakaway or break-in points. If a real root of Equation (6–14) lies on the root-locus portion of the real axis, then it is an actual breakaway or break-in point. If a real root of Equation (6–14) is not on the root-locus portion of the real axis, then this root corresponds to neither a breakaway point nor a break-in point.

If two roots $s=s_1$ and $s=-s_1$ of Equation (6–14) are a complex-conjugate pair and if it is not certain whether they are on root loci, then it is necessary to check the corresponding K value. If the value of K corresponding to a root $s=s_1$ of $dK/ds = 0$ is positive, point $s=s_1$ is an actual breakaway or break-in point. (Since K is assumed to be nonnegative, if the value of K thus obtained is negative, or a complex quantity, then point $s=s_1$ is neither a breakaway nor a break-in point.)

5. *Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero).* To sketch the root loci with reasonable accuracy, we must find the directions of the root loci near the complex poles and zeros. If a test point is chosen and moved in the very vicinity of a complex pole (or complex zero), the sum of the angular contributions from all other poles and zeros can be considered to remain the same. Therefore, the angle of departure (or angle of arrival) of the root locus from a complex pole (or at a complex zero) can be found by subtracting from 180° the sum of all the angles of vectors from all other poles and zeros to the complex pole (or complex zero) in question, with appropriate signs included.

Angle of departure from a complex pole = 180°
− (sum of the angles of vectors to a complex pole in question from other poles)
± (sum of the angles of vectors to a complex pole in question from zeros)

Angle of arrival at a complex zero = 180°
− (sum of the angles of vectors to a complex zero in question from other zeros)
± (sum of the angles of vectors to a complex zero in question from poles)

The angle of departure is shown in Figure 6–12.

6. *Find the points where the root loci may cross the imaginary axis.* The points where the root loci intersect the jv axis can be found easily by (a) use of Routh's stability criterion or (b) letting $s=jv$ in the characteristic equation, equating both the real part and the imaginary part to zero, and solving for v and K. The values of v thus found give the frequencies at which root loci cross the imaginary axis. The K value corresponding to each crossing frequency gives the gain at the crossing point.

7. *Taking a series of test points in the broad neighborhood of the origin of the s plane, sketch the root loci.* Determine the root loci in the broad neighborhood of the jv axis and the origin. The most important part of the root loci is on neither the real axis nor the asymptotes but is in the broad neighborhood of the jv axis and the origin. The shape

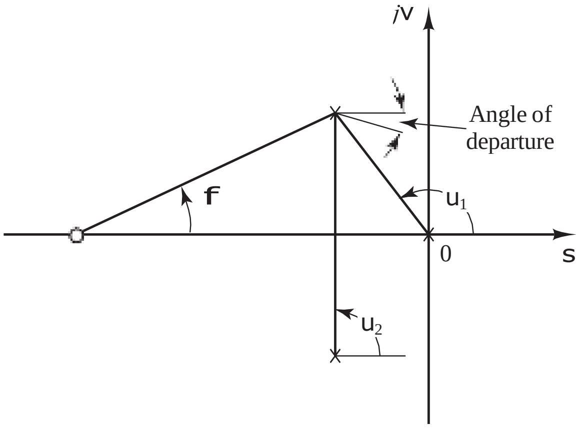


Figure 6–12 Construction of the root locus. [Angle of departure $= 180° - (u_1 + u_2) + f$.]

of the root loci in this important region in the s plane must be obtained with reasonable accuracy. (If accurate shape of the root loci is needed, MATLAB may be used rather than hand calculations of the exact shape of the root loci.)

8. *Determine closed-loop poles.* A particular point on each root-locus branch will be a closed-loop pole if the value of K at that point satisfies the magnitude condition. Conversely, the magnitude condition enables us to determine the value of the gain K at any specific root location on the locus. (If necessary, the root loci may be graduated in terms of K. The root loci are continuous with K.)

The value of K corresponding to any point s on a root locus can be obtained using the magnitude condition, or

$$K = \frac{\text{product of lengths between point } s \text{ to poles}}{\text{product of lengths between point } s \text{ to zeros}}$$

This value can be evaluated either graphically or analytically. (MATLAB can be used for graduating the root loci with K. See Section 6–3.)

If the gain K of the open-loop transfer function is given in the problem, then by applying the magnitude condition, we can find the correct locations of the closed-loop poles for a given K on each branch of the root loci by a trial-and-error approach or by use of MATLAB, which will be presented in Section 6–3.

Comments on the Root-Locus Plots. It is noted that the characteristic equation of the negative feedback control system whose open-loop transfer function is

$$G(s)H(s) = \frac{K(s^m + b_1 s^{m-1} + \cdots + b_m)}{s^n + a_1 s^{n-1} + \cdots + a_n} \qquad (n \geq m)$$

is an nth-degree algebraic equation in s. If the order of the numerator of $G(s)H(s)$ is lower than that of the denominator by two or more (which means that there are two or more zeros at infinity), then the coefficient a_1 is the negative sum of the roots of the equation and is independent of K. In such a case, if some of the roots move on the locus toward the left as K is increased, then the other roots must move toward the right as K is increased. This information is helpful in finding the general shape of the root loci.

It is also noted that a slight change in the pole–zero configuration may cause significant changes in the root-locus configurations. Figure 6–13 demonstrates the fact that a slight change in the location of a zero or pole will make the root-locus configuration look quite different.

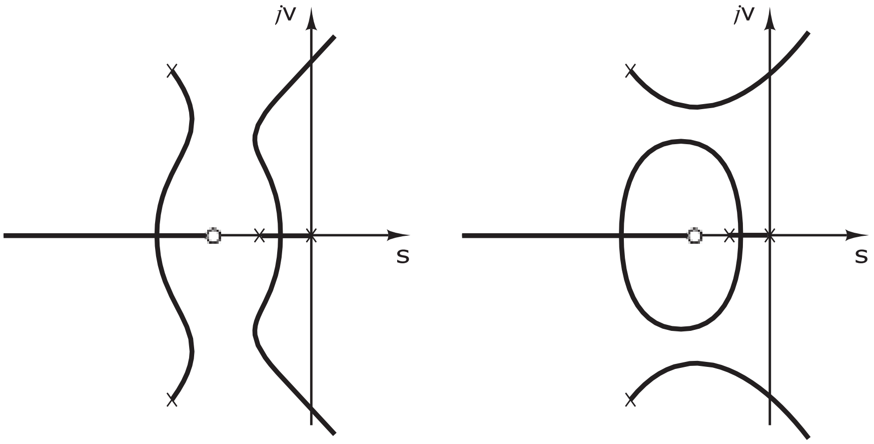


Figure 6–13
Root-locus plots.

A–6–1. Sketch the root loci for the system shown in Figure 6–64(a). (The gain K is assumed to be positive.) Observe that for small or large values of K the system is overdamped and for medium values of K it is underdamped.

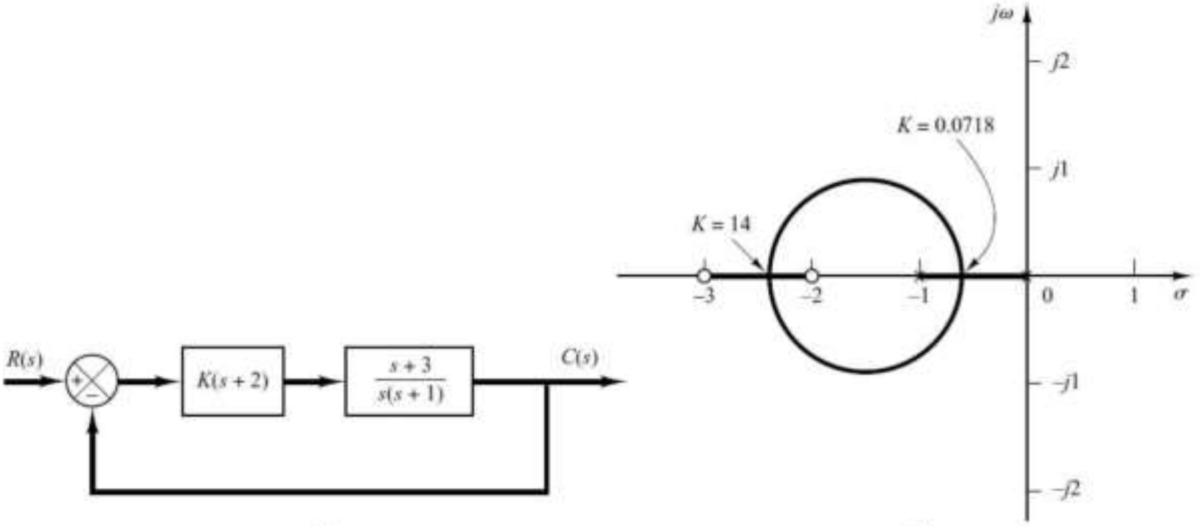
Solution. The procedure for plotting the root loci is as follows:

1. Locate the open-loop poles and zeros on the complex plane. Root loci exist on the negative real axis between 0 and −1 and between −2 and −3.
2. The number of open-loop poles and that of finite zeros are the same. This means that there are no asymptotes in the complex region of the s plane.
3. Determine the breakaway and break-in points. The characteristic equation for the system is

$$1 + \frac{K(s+2)(s+3)}{s(s+1)} = 0$$

or

$$K = -\frac{s(s+1)}{(s+2)(s+3)}$$



(a)

(b)

Figure 6–64
(a) Control system; (b) root-locus plot.

The breakaway and break-in points are determined from

$$\frac{dK}{ds} = -\frac{(2s+1)(s+2)(s+3) - s(s+1)(2s+5)}{\left[(s+2)(s+3)\right]^2}$$

$$= -\frac{4(s+0.634)(s+2.366)}{\left[(s+2)(s+3)\right]^2}$$

$$= 0$$

as follows:

$$s = -0.634, \qquad s = -2.366$$

Notice that both points are on root loci. Therefore, they are actual breakaway or break-in points. At point $s = -0.634$, the value of K is

$$K = -\frac{(-0.634)(0.366)}{(1.366)(2.366)} = 0.0718$$

Similarly, at $s = -2.366$,

$$K = -\frac{(-2.366)(-1.366)}{(-0.366)(0.634)} = 14$$

(Because point $s = -0.634$ lies between two poles, it is a breakaway point, and because point $s = -2.366$ lies between two zeros, it is a break-in point.)

4. Determine a sufficient number of points that satisfy the angle condition. (It can be found that the root loci involve a circle with center at -1.5 that passes through the breakaway and break-in points.) The root-locus plot for this system is shown in Figure 6–64(b).

Note that this system is stable for any positive value of K since all the root loci lie in the left-half s plane.

Small values of K ($0 < K < 0.0718$) correspond to an overdamped system. Medium values of K ($0.0718 < K < 14$) correspond to an underdamped system. Finally, large values of K ($14 < K$) correspond to an overdamped system. With a large value of K, the steady state can be reached in much shorter time than with a small value of K.

The value of K should be adjusted so that system performance is optimum according to a given performance index.