



Al-Mustaqbal University

Subject Name: Engineering Analysis and Numerical Methods

3rd Class, Second Semester

Subject Code: [Insert Subject Code Here]

Academic Year: 2025-2026

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Laplace Transform Theorem (Properties)

Theorem 1: Linearity

The Laplace transform is a linear transform, by which is meant that:

(1) *The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is*

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$

(2) *The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is*

$$L\{kf(t)\} = kL\{f(t)\}$$

Note: Two transforms must **not** be multiplied together to form the transform of a product of expressions – we shall see later that the product of two transforms is the transform of the convolution of two expressions.

Example

$$(a) L\{2e^{-t} + t\} = L\{2e^{-t}\} + L\{t\} = 2L\{e^{-t}\} + L\{t\} = \frac{2}{s+1} + \frac{1}{s^2} = \frac{2s^2 + s + 1}{s^2(s+1)}$$

$$(b) L\{2 \sin 3t + \cos 3t\} = 2L\{\sin 3t\} + L\{\cos 3t\} = 2 \cdot \frac{3}{s^2 + 9} + \frac{s}{s^2 + 9} = \frac{s + 6}{s^2 + 9}$$

$$(c) L\{4e^{2t} + 3 \cosh 4t\} = 4L\{e^{2t}\} + 3L\{\cosh 4t\} \\ = 4 \cdot \frac{1}{s-2} + 3 \cdot \frac{s}{s^2 - 16} = \frac{4}{s-2} + \frac{3s}{s^2 - 16} \\ = \frac{7s^2 - 6s - 64}{(s-2)(s^2 - 16)}$$

The working is straightforward. **for**

$$(d) L\{2 \sin 3t + 4 \sinh 3t\} = 2 \cdot \frac{3}{s^2 + 9} + 4 \cdot \frac{3}{s^2 - 9} \\ = \frac{6}{s^2 + 9} + \frac{12}{s^2 - 9} = \frac{18(s^2 + 3)}{s^4 - 81}$$

$$(e) L\{5e^{4t} + \cosh 2t\} = \frac{5}{s-4} + \frac{s}{s^2 - 4} = \frac{6s^2 - 4s - 20}{(s-4)(s^2 - 4)}$$

$$(f) L\{t^3 + 2t^2 - 4t + 1\} = \frac{3!}{s^4} + 2 \cdot \frac{2!}{s^3} - 4 \cdot \frac{1!}{s^2} + \frac{1}{s} \\ = \frac{1}{s^4} \{s^3 - 4s^2 + 4s + 6\}$$

Theorem 2 The first shift theorem

The first shift theorem states that if $L\{f(t)\} = F(s)$ then

$$L\{e^{-at}f(t)\} = F(s+a)$$

$$\text{Because } L\{e^{-at}f(t)\} = \int_{t=0}^{\infty} e^{-at}f(t)e^{-st} dt = \int_{t=0}^{\infty} f(t)e^{-(s+a)t} dt = F(s+a)$$

$$\text{That is } L\{e^{-at}f(t)\} = F(s+a)$$

The transform $L\{e^{-at}f(t)\}$ is thus the same as $L\{f(t)\}$ with s everywhere in the result replaced by $(s+a)$.

$$\text{For example } L\{\sin 2t\} = \frac{2}{s^2 + 4}$$

$$\text{then } L\{e^{-3t} \sin 2t\} = \frac{2}{(s+3)^2 + 4} = \frac{2}{s^2 + 6s + 13}$$

Similarly, $L\{t^2\} = \frac{2}{s^3}$. $\therefore L\{t^2 e^{4t}\}$ is the same with s replaced by $(s-4)$.

$$\therefore L\{t^2 e^{4t}\} = \frac{2}{(s-4)^3}$$

Exercise

Determine the following.

1. $L\{e^{-2t} \cosh 3t\}$
2. $L\{2e^{3t} \sin 3t\}$
3. $L\{4te^{-t}\}$
4. $L\{e^{2t} \cos t\}$
5. $L\{e^{3t} \sinh 2t\}$
6. $L\{t^3 e^{-4t}\}$

$$1. L\{\cosh 3t\} = \frac{s}{s^2 - 9} \quad \therefore L\{e^{-2t} \cosh 3t\} = \frac{s+2}{(s+2)^2 - 9} \\ = \frac{s+2}{s^2 + 4s - 5}$$

$$2. L\{\sin 3t\} = \frac{3}{s^2 + 9} \quad \therefore L\{2e^{3t} \sin 3t\} = \frac{6}{(s-3)^2 + 9} \\ = \frac{6}{s^2 - 6s + 18}$$

$$3. L\{4t\} = 4 \cdot \frac{1}{s^2} \quad \therefore L\{4te^{-t}\} = \frac{4}{(s+1)^2}$$

$$4. L\{\cos t\} = \frac{s}{s^2 + 1} \quad \therefore L\{e^{2t} \cos t\} = \frac{s-2}{(s-2)^2 + 1} \\ = \frac{s-2}{s^2 - 4s + 5}$$

$$5. L\{\sinh 2t\} = \frac{2}{s^2 - 4} \quad \therefore L\{e^{3t} \sinh 2t\} = \frac{2}{(s-3)^2 - 4} \\ = \frac{2}{s^2 - 6s + 5}$$

$$6. L\{t^3\} = \frac{3!}{s^4} \quad \therefore L\{t^3 e^{-4t}\} = \frac{6}{(s+4)^4}$$

Theorem 3 Multiplying by t and t^n

If $L\{f(t)\} = F(s)$ then $L\{tf(t)\} = -F'(s)$

$$\begin{aligned}\text{Because } L\{tf(t)\} &= \int_{t=0}^{\infty} tf(t)e^{-st} dt = \int_{t=0}^{\infty} f(t) \left(-\frac{de^{-st}}{ds}\right) dt \\ &= -\frac{d}{ds} \int_{t=0}^{\infty} f(t)e^{-st} dt = -F'(s)\end{aligned}$$

For example, $L\{\sin 2t\} = \frac{2}{s^2 + 4}$

$$\therefore L\{t \sin 2t\} = -\frac{d}{ds} \left(\frac{2}{s^2 + 4} \right) = \frac{4s}{(s^2 + 4)^2}$$

and similarly, $L\{t \cosh 3t\} = -\frac{d}{ds} \left(\frac{s}{s^2 - 9} \right) = -\frac{(s^2 - 9) - s(2s)}{(s^2 - 9)^2}$
$$= \frac{s^2 + 9}{(s^2 - 9)^2}$$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \{F(s)\}$$

Example

$$\mathcal{L}\{t^2 \sin(t)\}$$

Here, $f(t) = \sin(t)$ then $F(s) = \frac{1}{s^2 + 1}$

$$\mathcal{L}\{t^2 \sin(t)\} = (-1)^2 \frac{d^2}{ds^2} \left\{ \frac{1}{s^2 + 1} \right\}$$

$$\frac{d}{ds} \left\{ \frac{1}{s^2 + 1} \right\} = \frac{-2s}{(s^2 + 1)^2}$$

$$\frac{d^2}{ds^2} \left\{ \frac{1}{s^2 + 1} \right\} = \frac{(s^2 + 1)^2 \times (-2) + 2s \times 2(s^2 + 1)(2s)}{(s^2 + 1)^4} = \frac{(s^2 + 1)[(-2)(s^2 + 1) + 8s^2]}{(s^2 + 1)^4}$$

$$= \frac{6s^2 - 2}{(s^2 + 1)^3}$$

Theorem 4 Division by t

If $F(s) = \mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(u) du$$

Example

$$\mathcal{L}\left\{\frac{\sin(t)}{t}\right\}$$

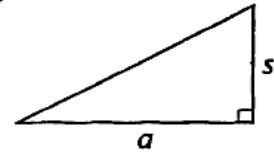
Here, $f(t) = \sin(t)$ then $F(s) = \frac{1}{s^2 + 1}$ and

$$\mathcal{L}\left\{\frac{\sin(t)}{t}\right\} = \int_s^{\infty} \frac{1}{u^2 + 1} du = \tan^{-1}(u) \Big|_s^{\infty} = \frac{\pi}{2} - \tan^{-1}(s) = \tan^{-1}\left(\frac{1}{s}\right)$$

$$= \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right)$$

$$= \arctan\left(\frac{a}{s}\right)$$

Notice that $\arctan\left(\frac{a}{s}\right) + \arctan\left(\frac{s}{a}\right) = \frac{\pi}{2}$, as can be seen from the figure



Example 2

Determine $L\left\{\frac{1 - \cos 2t}{t}\right\}$

$$L\{1 - \cos 2t\} = \frac{1}{s} - \frac{s}{s^2 + 4}$$

Then, by Theorem 4

$$\begin{aligned} L\left\{\frac{1 - \cos 2t}{t}\right\} &= \int_{\sigma=s}^{\infty} \left\{\frac{1}{\sigma} - \frac{\sigma}{\sigma^2 + 4}\right\} d\sigma \\ &= \left[\ln \sigma - \frac{1}{2} \ln(\sigma^2 + 4)\right]_{\sigma=s}^{\infty} = \frac{1}{2} \left[\ln\left(\frac{\sigma^2}{\sigma^2 + 4}\right)\right]_{\sigma=s}^{\infty} \end{aligned}$$

When $\sigma \rightarrow \infty$, $\ln\left(\frac{\sigma^2}{\sigma^2 + 4}\right) \rightarrow \ln 1 = 0$

$$\begin{aligned}\text{Therefore, } L\left\{\frac{1 - \cos 2t}{t}\right\} &= -\frac{1}{2} \ln\left(\frac{s^2}{s^2 + 4}\right) = \ln\left(\frac{s^2}{s^2 + 4}\right)^{-1/2} \\ &= \ln \sqrt{\frac{s^2 + 4}{s^2}}\end{aligned}$$