Al-Mustaqbal University

College of Engineering and Technology

Department of Biomedical Engineering

Stage: Second

Electric Circuits II

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Lecture (4): Phasor representation

CHAPTER NINE

PHASORS AND THE BASIC ELEMENTS

9.1 INTRODUCTION

The response of the basic R, L, and C elements to a sinusoidal voltage and current will be examined in this chapter, with special note of how frequency will affect the "opposing" characteristic of each element. Phasor notation will then be introduced to establish a method of analysis that permits a direct correspondence with a number of the methods, theorems, and concepts introduced in the dc chapters.

9.2 PHASORS

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions.

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

A complex number z can be written in rectangular form as

$$\mathbf{z = x + jy} \tag{9.1a}$$

where $j = \sqrt{-1}$; x is the real part of z; y is the imaginary part of z.

The complex number z can also be written in polar or exponential form as

$$\mathbf{z = r \angle \varphi = re^{j\varphi}} \tag{9.1b}$$

where r is the magnitude of z, and φ is the phase of z. We notice that z can be represented in three ways:

$\mathbf{z = x + jy}$	Rectangular form	
$\mathbf{z = r \angle \varphi}$	Polar form	
$\mathbf{z = r e^{j\varphi}}$	Exponential form	(9.2)

The relationship between the rectangular form and the polar form is shown below, where the x axis represents the real part and the y axis represents the imaginary part of a complex number. Given x and y, we can get r and φ as

$$r = \sqrt{x^2 + y^2} \ , \ \varphi = \tan^{-1}\frac{y}{x} \tag{9.3a}$$

On the other hand, if we know r and φ, we can obtain x and y as

$$\mathbf{x = r \cos \varphi, \ \ y = r \sin \varphi} \tag{9.3b}$$

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Thus, z may be written as

$$\mathbf{z = x + jy = r \angle\varphi = r (\cos\varphi + j \sin\varphi)} \quad (9.4)$$

Addition and subtraction of complex numbers are better performed in rectangular form; multiplication and division are better done in polar form. Given the complex numbers

$z = x + jy = r\ \varphi$, $z_1 = x_1 + jy_1 = r_1 \angle\varphi1$, $z_2 = x_2 + jy_2 = r_2 \angle\varphi1$

The following operations are important.

Addition: $z_1 + z_2 = (x_1 + x_2) + j\,(y_1 + y_2)$ (9.5a)

Subtraction: $z_1 - z_2 = (x_1 - x_2) + j\,(y_1 - y_2)$ (9.5b)

Multiplication: $z_1z_2 = r_1r_2 \angle\varphi1 + \varphi2$ (9.5c)

Division: $\frac{z1}{z2} = \frac{r1}{r2} \angle\varphi1 - \varphi2$ (9.5d)

Reciprocal: $\frac{1}{z} = \frac{1}{r} \angle\varphi1$ (9.5e)

Square Root: $\sqrt{z} = \sqrt{r} \angle\varphi/2$ (9.5f)

Complex Conjugate: $z^* = x - jy = r \angle{-\varphi} = r\, e^{-j\varphi}$ (9.5g)

Note that from Eq. (9.5e),

$$1/j = -j \quad (9.5h)$$

The idea of phasor representation is based on Euler's identity. In general,

$$e^{\pm j\varphi} = \cos\varphi \pm j \sin\varphi \quad (9.6)$$

$$\cos\varphi = \mathrm{Re}(e^{j\varphi}) \quad (9.7a)$$

$$\sin\varphi = \mathrm{Im}(e^{j\varphi}) \quad (9.7b)$$

where Re and Im stand for the real part of and the imaginary part of.

Given a sinusoid $v(t) = V_m \cos(\omega t + \varphi)$, we use Eq. (9.7a) to express v(t) as

$$\mathbf{v(t)= V_m \cos(\omega t + \varphi) = Re(V_m e^{j(\omega t+\varphi)})= Re(V_m e^{j\varphi} e^{j\omega t})} \quad (9.8)$$

Thus,

$$\mathbf{v(t) = Re(V e^{j\omega t})} \quad (9.9)$$

where

$$\mathbf{V = V_m e^{j\varphi} = V_m \angle\varphi} \quad (9.10)$$

V is thus the phasor representation of the sinusoid v(t), as we said earlier. In other words, a phasor is a complex representation of the magnitude and phase of a sinusoid.

One way of looking at Eqs. (9.9) and (9.10) is to consider the plot in Fig. 9.1(a)and (b) of the sinor $V e^{j\omega t} = V_m e^{j(\omega t+\varphi)}$ on the complex plane. As time increases, the sinor rotates on a

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circle of radius V_m at an angular velocity ω in the counterclockwise direction, as shown in. In other words, the entire complex plane is rotating at an angular velocity of ω. We may regard v(t) as the projection of the sinor Ve jωt on the real axis, as shown in Fig. 9.7(b). The value of the sinor at time t = 0 is the phasor V of the sinusoid v(t). The sinor may be regarded as a rotating phasor.

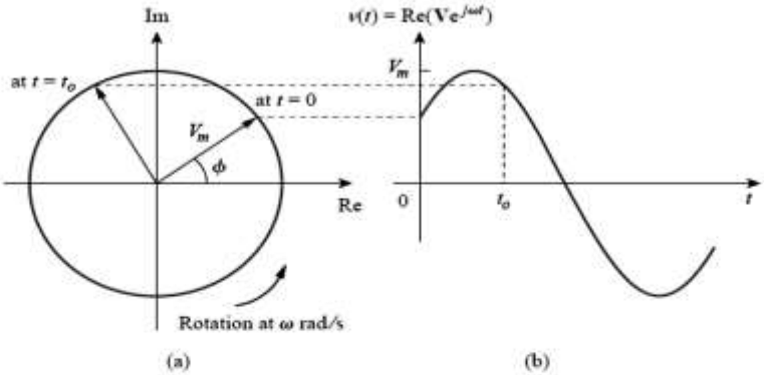

Figure 9.1 Representation of $\mathbf{V}e^{j\omega t}$: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

For example, phasors $V = V_m \angle\varphi$ and $I = I_m \angle-\theta$ are graphically represented in Fig. 9.2. Such a graphical representation of phasors is known as a phasor diagram.

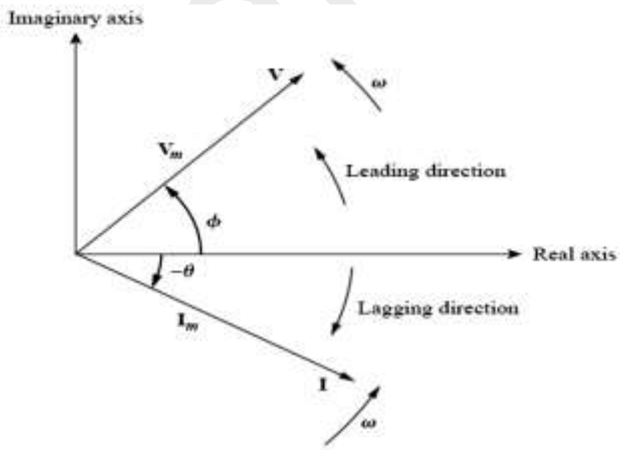

Figure 9.2 A phasor diagram showing V = *Vm ∠φ* and I = *Im ∠−θ.*

By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$v(t) = V_m \cos(\omega t + \varphi) \quad \Longleftrightarrow \quad \mathbf{V} = V_m \angle \varphi \qquad (9.11)$$

(Time-domain representation) (Phasor-domain representation)

Note that in Eq. (9.11) the frequency (or time) factor e jωt is suppressed, and the frequency is not explicitly shown in the phasor-domain representation because ω is constant. However, the response depends on ω. For this reason, the phasor domain is also known as the frequency domain.

From Eqs. (9.9) and (9.10), $v(t)= Re(V e^{j\omega t}) = V_m \cos(\omega t + \varphi)$, so that

$$\frac{dv}{dt} = -\omega V_m \sin(\omega t + \varphi) = \omega V_m \cos(\omega t + \varphi + 90^\circ) = Re(\omega V_m e^{j\omega t} e^{j\varphi} e^{j90^\circ}) = Re(j\omega V e^{j\omega t}) \quad (9.12)$$

This shows that the derivative v(t) is transformed to the phasor domain as jωV

$$dv/dt \ \text{(Timedomain)} \quad \Longleftrightarrow \quad j\omega V \ \text{(Phasor domain)} \quad (9.13)$$

Similarly, the integral of v(t) is transformed to the phasor domain as V/jω

$$\int v\,dt \ \text{(Timedomain)} \quad \Longleftrightarrow \quad V/j\omega \ \text{(Phasor domain)} \quad (9.14)$$

Example 9.1: Transform these sinusoids to phasors:

(a) $v =- 4 \sin(30t + 50^\circ)$

(b) $i = 6 \cos(50t - 40^\circ)$

Solution: (a) Since $- \sin A = \cos(A + 90^\circ)$,

$$v =- 4 \sin(30t + 50^\circ) = 4 \cos(30t + 50^\circ + 90^\circ) = 4 \cos(30t + 140^\circ)$$

The phasor form of v is

$$V = 4 \angle 140^\circ$$

(b) $i = 6 \cos(50t - 40^\circ)$ has the phasor

$$I = 6 \angle -40^\circ$$

Practice problem 9.1: Express these sinusoids as phasors:

(a) $v =-7 \cos(2t + 40^\circ)$

(b) $i = 4 \sin(10t + 10^\circ)$

Answer: (a) $V = 7 \angle 220^\circ$, (b) $I = 4 \angle -80^\circ$.

Example 9.2: Find the sinusoids represented by these phasors:

(a) $V = j8e^{-j20^\circ}$

(b) $I =- 3 + j4$

Solution:

(a) Since $j = 1\angle 90^\circ$,

$$V = j8 \angle{-20^\circ} = (1 \angle 90^\circ)\,(8 \angle{-20^\circ})$$

$$= 8 \angle(90^\circ - 20^\circ) = 8 \angle 70^\circ \text{ V}$$

Converting this to the time domain gives

$$v(t) = 8 \cos(\omega t + 70^\circ) \text{ V}$$

(b) $I = -3 + j4 = 5 \angle 126.87^\circ$. Transforming this to the time domain gives

$$i(t) = 5 \cos(\omega t + 126.87^\circ) \text{ A}$$

Practice problem 9.2: Find the sinusoids corresponding to these phasors:

(a) $V = -10 \angle 30^\circ$

(b) $I = j(5 - j12)$

Answer: (a) $v(t) = 10 \cos(\omega t + 210^\circ)$, (b) $i(t) = 13 \cos(\omega t + 22.62^\circ)$.

9.3 PHASOR RELATIONSHIPS FOR CIRCUIT ELEMENTS

Now that we know how to represent a voltage or current in the phasor or frequency domain, one may legitimately ask how we apply this to circuits involving the passive elements R, L, and C. What we need to do is to transform the voltage-current relationship from the time domain to the frequency domain for each element. Again, we will assume the passive sign convention.

Resistor

We begin with the resistor. If the current through a resistor R is $i = I_m \cos(\omega t + \varphi)$, the voltage across it is given by Ohm's law as

$$\mathbf{v = iR = R\, I_m \cos(\omega t + \varphi)} \qquad (9.15)$$

The phasor form of this voltage is

$$\mathbf{V = R\, I_m \angle \varphi} \qquad (9.16)$$

But the phasor representation of the current is $I = I_m \angle \varphi$. Hence,

$$\mathbf{V = RI} \qquad (9.17)$$

showing that the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain. Figure 9.3 illustrates the voltage-current relations of a resistor. We should note from Eq. (9.17) that voltage and current are in phase, as illustrated in the phasor diagram in Fig. 9.4.

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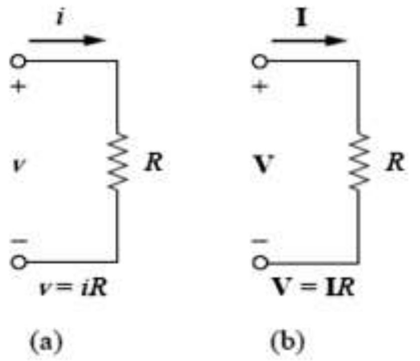

Figure 9.3 Voltage-current relations for a resistor in the: (a) time domain, (b) frequency domain.

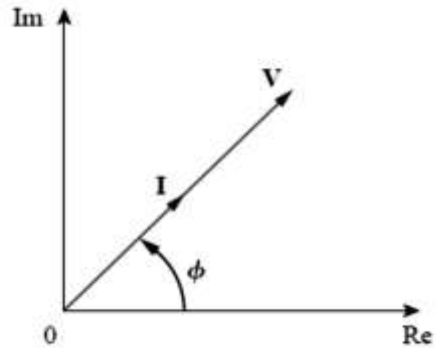

Figure 9.4 Phasor diagram for the resistor.

Inductor

For the inductor L, assume the current through it is $i = I_m \cos(\omega t + \varphi)$. The voltage across the inductor is

$$v = L\, di/dt = -\omega L I_m \sin(\omega t + \varphi) \quad (9.18)$$

Recall that $-\sin A = \cos(A + 90^\circ)$. We can write the voltage as

$$v = \omega L I_m \cos(\omega t + \varphi + 90^\circ) \quad (9.19)$$

which transforms to the phasor

$$V = \omega L I_m e^{j(\varphi+90^\circ)} = \omega L I_m e^{j\varphi} e^{j90^\circ} = \omega L I_m \angle\varphi\, e^{j90^\circ} \quad (9.20)$$

But $I_m \varphi = I$, and from Eq. (9.19), $e^{j90^\circ} = j$. Thus,

$$V = j\omega L I \quad (9.21)$$

showing that the voltage has a magnitude of $\omega L I_m$ and a phase of $\varphi + 90^\circ$. The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°. Figure 9.5 shows the voltage-current relations for the inductor. Figure 9.6 shows the phasor diagram.

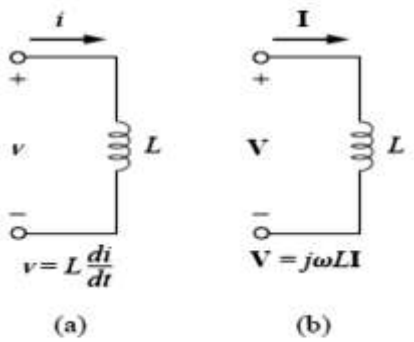

Figure 9.5 Voltage-current relations for an inductor in the: (a) time domain, (b) frequency domain.

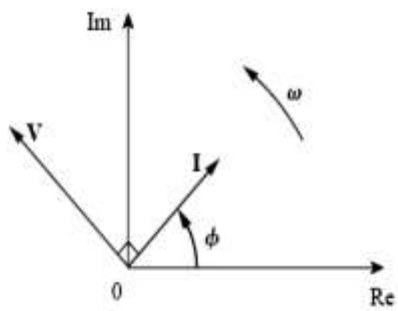

Figure 9.6 Phasor diagram for the inductor; I lags V.

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Capacitor

For the capacitor C, assume the voltage across it is $v = V_m \cos(\omega t + \varphi)$. The current through the capacitor is

i = C dv/dt (9.22)

By following the same steps as we took for the inductor or by applying Eq. (9.13) on Eq. (9.22), we obtain

I = jωCV ⇒ V = I / jωC (9.23)

showing that the current and voltage are 90° out of phase. To be specific, the current leads the voltage by 90°. Figure 9.7 shows the voltage-current relations for the capacitor; Fig. 9.8 gives the phasor diagram.

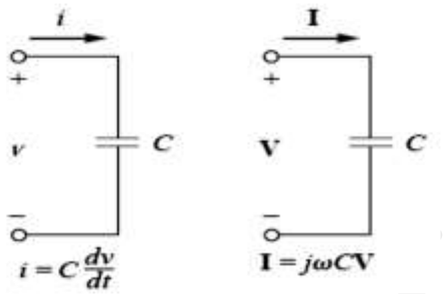

Figure 9.7 Voltage-current relations for a capacitor in the: (a) time domain, (b) frequency domain.

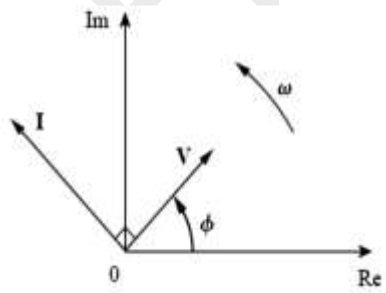

Figure 9.8 Phasor diagram for the capacitor; I leads V.

Example 9.3: The voltage v = 12 cos(60t + 45°) is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Solution: For the inductor, V = jωLI, where ω = 60 rad/s and V = 12 ∠45° V. Hence

$$I = \frac{V}{j\omega L} = \frac{12\angle 45^\circ}{j60 \times 0.1} = \frac{12\angle 45^\circ}{6\angle 90^\circ} = 2\angle -45^\circ\ A$$

Converting this to the time domain,

i(t)= 2 cos(60t − 45°) A

Practice problem 9.3: If voltage v = 6 cos(100t − 30°) is applied to a 50 µF capacitor, calculate the current through the capacitor.

Answer: 30 cos(100t + 60°) mA.

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