Al-Mustaqbal University
Department (Medical Instrumentation Techniques Engineering)
Class (3rd)
Subject (Electromagnetic Fields)
Lecturer (Dr. Osamah Jaber Ghayyib)
5th Semester – Lect. (Gauss's law)

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CONTENTS

LEARNING OBJECTIVES

By the end of this lecture, students will be able to:

1. Explain the concept of **electric flux** and **electric flux density** and their relationship to electric field intensity.

2. Describe **Faraday's experiments** on electric displacement and their role in understanding dielectric materials.

3. State and interpret **Gauss's Law** in both integral and differential forms.

4. Apply **Gauss's Law** to determine electric fields for charge distributions with **spherical, cylindrical, and planar symmetry**.

5. Relate Gauss's Law to **Coulomb's Law** and recognize its importance as one of **Maxwell's Equations**.

6. Identify real-world applications of Gauss's Law in **electrical and biomedical systems**, such as capacitive sensors and electric field shielding in medical instruments.

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1   INTRODUCTION

The study of electric fields and their interaction with matter forms the foundation of electromagnetic theory. One of the key principles that describe these interactions is **Gauss's Law**, a fundamental relationship between electric charge and the electric flux it produces.

Historically, the development of Gauss's Law is linked to the pioneering work of **Michael Faraday** in the 1830s, whose experiments demonstrated how electric fields behave in the presence of dielectric materials. These findings paved the way for a deeper understanding of electric flux and the mathematical formulation introduced by **Carl Friedrich Gauss**, which provides a powerful method to calculate electric fields for highly symmetrical charge distributions.

This lecture introduces the physical concepts behind Gauss's Law, its mathematical derivation, and its applications in practical engineering problems

2   FARADAY'S EXPERIMENTS ON ELECTRIC DISPLACEMENT

About 1837, the director of the Royal Society in London, Michael Faraday, became very interested in static electric fields and the effect of various insulating materials on these fields. This problem had been bothering him during the previous ten years when he was experimenting in his now-famous work on induced electromotive force. With that subject completed, he had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulating material (or dielectric material, or simply dielectric) that would occupy the entire volume between the concentric spheres. His experiment, then, consisted essentially of the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge.

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2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.

3. The outer sphere was discharged by connecting it momentarily to ground.

4. The outer sphere was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured

A Cage of His Own
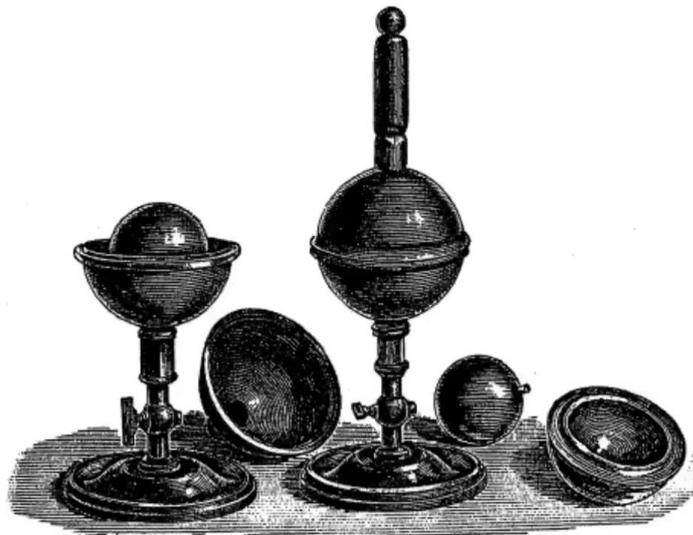
The nested-sphere apparatus Faraday used to study electrostatic induction.

Figure 1: Nested-sphere apparatus of Faraday Experiment.

Faraday found that the total charge on the outer sphere was equal in magnitude to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres.

If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment:

$$\Psi = Q \tag{1}$$

and the electric flux Ψ is measured in coulombs.

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3 ELECTRIC FLUX DENSITY

Consider an inner sphere of radius a and an outer sphere of radius b, with charges of Q and −Q, respectively (Figure 2). The paths of electric flux Ψ extending from the inner sphere to the outer sphere are indicated by the symmetrically distributed streamlines drawn radially from one sphere to the other.

At the surface of the inner sphere, Ψ coulombs of electric flux are produced by the charge $Q \, (= \Psi)$ coulombs distributed uniformly over a surface having an area of $4\pi a^2 \mathrm{m}^2$.

The density of the flux at this surface is $\Psi/4\pi a^2$ or $Q/4\pi a^2 \mathrm{C/m}^2$, and this is an important new quantity.
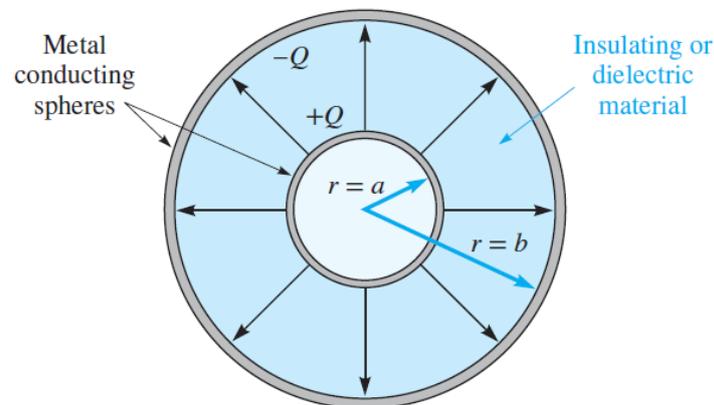

Figure 2: The electric flux in the region between a pair of charged concentric spheres.

The electric flux density **D** is a vector field and defined in free space and for a point charge as follows:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \qquad (2)$$

Since the electric field intensity of a point charge in free space:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r \qquad (3)$$

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Therefore,

$$\mathbf{D} = \varepsilon_o \mathbf{E} \qquad \text{(free space only)} \tag{4}$$

EXAMPLE 1 : Determine **D** at (4,0,3) if there is a point charge -5π mC at (4,0,0) and a line charge 3π mC/m along the y-axis. Where the Electric field intensity of a line charge is $\mathbf{E}_L = \dfrac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho$.and ρ is the distance between the line and the point.

Solution

(a)   Let $\mathbf{D} = \mathbf{D}_Q + \mathbf{D}_L$, where $\mathbf{D}_Q$ and $\mathbf{D}_L$ are flux densities due to the point charge and line charge, respectively,
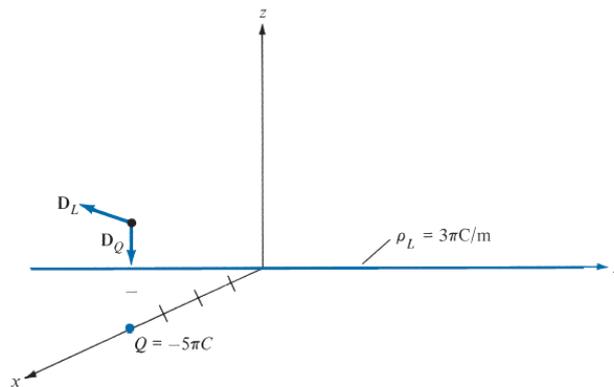
$$\mathbf{D}_Q = \varepsilon_o \mathbf{E} = \frac{Q}{4\pi R^2}\mathbf{a}_R = \frac{Q(\mathbf{r} - \mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} - \mathbf{r}' = (4,0,3) - (4,0,0) = (0,0,3)$. Hence,

$$\mathbf{D}_Q = \frac{-5\pi \cdot 10^{-3}(0,0,3)}{4\pi|(0,0,3)|^3} = -0.139\,\mathbf{a}_z\,m\text{C/m}^2$$

Also

$$\mathbf{D}_L = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho$$

In this case

$$\mathbf{a}_\rho = \frac{(4,0,3) - (0,0,0)}{|(4,0,3) - (0,0,0)|} = \frac{(4,0,3)}{5}$$
$$\rho = |(4,0,3) - (0,0,0)| = 5$$

Hence,

$$\mathbf{D}_L = \frac{3\pi}{2\pi(25)}(4\mathbf{a}_x + 3\mathbf{a}_z) = 0.24\mathbf{a}_x + 0.18\mathbf{a}_z\,\,m\text{C/m}^2$$

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Thus

$$\mathbf{D} = \mathbf{D}_Q + \mathbf{D}_L$$
$$= 240\mathbf{a}_x + 41.1\mathbf{a}_z \ \mu C/m^2$$

4 GAUSS'S LAW

Gauss's[5] law constitutes one of the fundamental laws of electromagnetism.

Gauss's law states that the total electric flux c through any *closed* surface is equal to the total charge enclosed by that surface

Since, $\Psi = Q_{enc}$ that is,

$$\Psi = \oint_S d\boldsymbol{\Psi} = \oint_S \mathbf{D} \cdot d\mathbf{S}$$
$$= \text{total charge enclosed } Q = \int_v \rho_v dv \tag{5}$$

Or

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \boldsymbol{\rho}_v dv \tag{6}$$

A mathematical statement meaning of (6) (7)simply that the total electric flux through any closed surface is equal to the charge enclosed.

By applying divergence theorem to the middle term in (6), we have:

$$\oint_S \boldsymbol{D} \cdot d\boldsymbol{S} = \int_v \nabla \cdot \boldsymbol{D} \ dv \tag{7}$$

Comparing the two volume integrals we conclude that:

$$\rho_v = \nabla \cdot \mathbf{D} \tag{8}$$

Which is the **first** of the four **Maxwell's equations** to be derived.

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5 APPLICATION OF GAUSS LAW

The procedure for applying Gauss's law to calculate the electric field involves first knowing whether symmetry exists. Once it has been found that symmetric charge distribution exists, we construct a mathematical closed surface (known as a Gaussian surface). The surface is chosen such that **D** is normal or tangential to the Gaussian surface. When **D** is normal to the surface, $\mathbf{D} \cdot d\mathbf{S} = D\,dS$ because **D** is perpendicular to the surface. When **D** is tangential to the surface, $\mathbf{D} \cdot d\mathbf{S} = 0$.

Thus, we must choose a surface that has some of the symmetry exhibited by the charge distribution. The choice of an appropriate Gaussian surface, where there is symmetry in the charge distribution comes from intuitive reasoning and a slight degree of maturity in the application of Coulomb's law. We shall now apply these basic ideas to the following cases:

5.1 A. POINT CHARGE

Suppose a point charge Q is located at the origin. To determine **D** at a point P, it is easy to see that choosing a spherical surface containing P will satisfy symmetry conditions. Thus, a spherical surface centered at the origin is the Gaussian surface in this case and is shown in Figure 3.
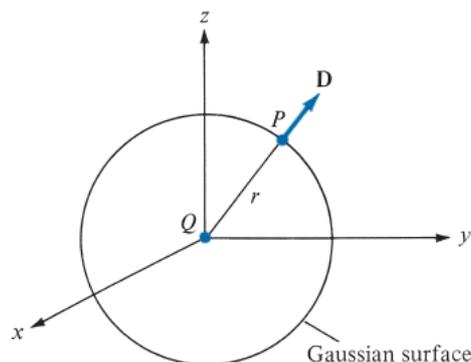

Figure 3: Gaussian surface about a point charge.

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Since **D** is everywhere normal to the Gaussian surface, that is, $\mathbf{D} = D_r \mathbf{a}_r$, applying Gauss's law ($\Psi = Q_{enc}$) gives:

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} = D_r \oint_S dS = D_r 4\pi r^2 \tag{9}$$

Thus,

$$\mathbf{D} = \frac{Q}{4\pi r^2} \mathbf{a}_r \tag{10}$$

5.2 INFINITE LINE CHARGE

Suppose the infinite line of uniform charge ρ_L C/m lies along the z-axis. To determine **D** at a point P, we choose a cylindrical surface containing P to satisfy the symmetry condition as shown in Figure 4.14. The electric flux density **D** is constant on and normal to the cylindrical Gaussian surface; that is, $\mathbf{D} = D_\rho \mathbf{a}_\rho$.
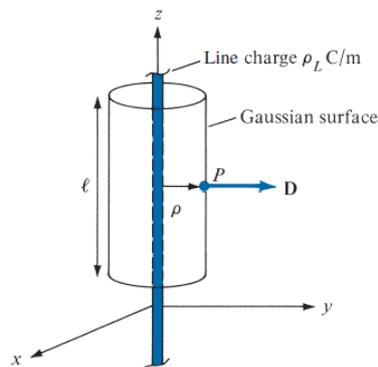

Figure 4: Gaussian surface about an infinite line charge.

If we apply Gauss's law to an arbitrary length ℓ of the line:

$$\rho_L \ell = Q = \int_S \mathbf{D} \cdot d\mathbf{S} = D_\rho \int_S dS = D_\rho 2\pi\rho\ell \tag{11}$$

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where $\int_S dS = 2\pi\rho\ell$ is the surface area of the Gaussian surface. Note that $\int \mathbf{D} \cdot d\mathbf{S}$ evaluated on the top and bottom surfaces of the cylinder is zero, since $\mathbf{D}$ has no z-component; that means that $\mathbf{D}$ is tangential to those surfaces. Thus:

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho \qquad (12)$$

5.3 INFINITE SHEET OF CHARGE

Consider an infinite sheet of uniform charge $\rho_S\,C/m^2$ lying on the $z = 0$ plane. To determine $\mathbf{D}$ at point P, we choose a rectangular box that is cut symmetrically by the sheet of charge and has two of its faces parallel to the sheet as shown in Figure 5. As $\mathbf{D}$ is normal to the sheet, $\mathbf{D} = D_z\mathbf{a}_z$, and applying Gauss's law gives:

$$\rho_S \int_S dS = Q = \int_S \mathbf{D} \cdot d\mathbf{S} = D_z\left[\int_{top} dS + \int_{bottom} dS\right] \qquad (13)$$
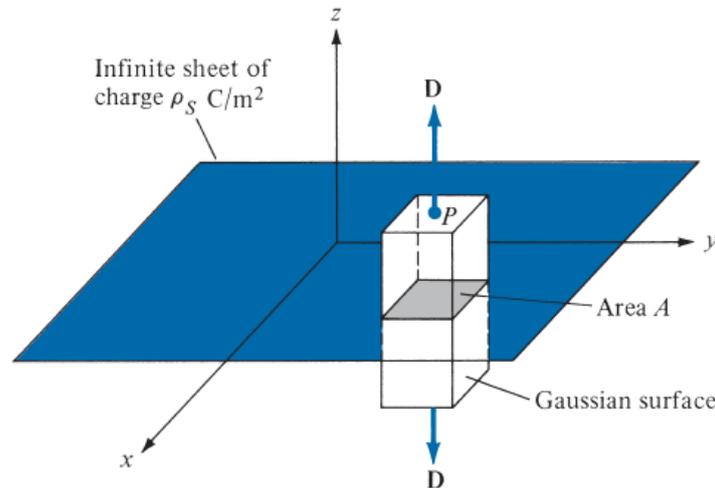

Figure 5: Gaussian surface about an infinite line sheet of charge.

Note that $\mathbf{D} \cdot d\mathbf{S}$ evaluated on the sides of the box is zero because $\mathbf{D}$ has no components along $\mathbf{a}_x$ and $\mathbf{a}_y$. If the top and bottom area of the box each has area A, (13) becomes $\rho_S A = D_z(A + A)$ and thus:

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$$\mathbf{D} = \frac{\rho_S}{2}\mathbf{a}_z \tag{14}$$

EXAMPLE 2 : Given that $\mathbf{D} = z\rho\cos^2\phi\,\mathbf{a}_z\,\text{C/m}^2$, calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1 m with $-2 \leq z \leq 2$ m

Solution
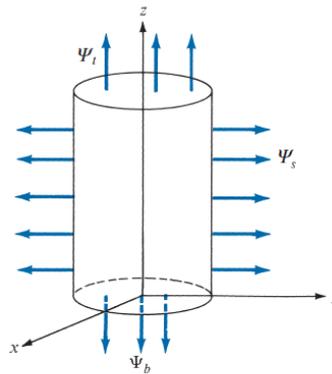
$$\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial D_z}{\partial z} = \rho\cos^2\phi$$

At $(1, \pi/4, 3)$, $\rho_v = 1 \cdot \cos^2(\pi/4) = 0.5\text{C/m}^3$.

The total charge enclosed by the cylinder can be found in two different ways.

Method 1: This method is based directly on the definition of the total volume charge.

$$Q = \int_v \rho_v dv = \int_v \rho\cos^2\phi\rho d\phi d\rho dz$$
$$= \int_{z=-2}^{2} dz \int_{\phi=0}^{2\pi} \cos^2\phi d\phi \int_{\rho=0}^{1} \rho^2 d\rho = 4(\pi)(1/3)$$
$$= \frac{4\pi}{3}\text{C}$$

Method 2: Alternatively, we can use Gauss's law

$$Q = \Psi = \int_S \mathbf{D} \cdot d\mathbf{S} = \left[\int_S + \int_t + \int_b\right]\mathbf{D} \cdot d\mathbf{S}$$
$$= \Psi_s + \Psi_t + \Psi_b$$

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where Ψ_s, Ψ_t, and Ψ_b are the flux through the sides (curved surface), the top surface, and the bottom surface of the cylinder, respectively. Since **D** does not have component along $\mathbf{a}_\rho$, $\Psi_s = 0$, for Ψ_t, $d\mathbf{S} = \rho d\phi d\rho \mathbf{a}_z$ so

$$\Psi_t = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} z\rho\cos^2\phi\rho d\phi d\rho \bigg|_{z=2} = 2\int_0^1 \rho^2 d\rho \int_0^{2\pi} \cos^2\phi d\phi$$

$$= 2\left(\frac{1}{3}\right)\pi = \frac{2\pi}{3}$$

and for $\boldsymbol{\Psi}_b$, $d\mathbf{S} = -\rho d\phi d\rho \mathbf{a}_z$, so

$$\Psi_b = -\int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} z\rho\cos^2\phi\rho d\phi d\rho \bigg|_{z=-2} = 2\int_0^1 \rho^2 d\rho \int_0^{2\pi} \cos^2\phi d\phi$$

$$= \frac{2\pi}{3}$$

Thus

$$Q = \Psi = 0 + \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3} \text{ C}$$

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