Integrals of inverse trigonometric functions:

1) $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c = -\cos^{-1}\frac{u}{a} + c \quad ; \quad \forall u^2 < a^2$
2) $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c$
3) $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c = -\frac{1}{a}\csc^{-1}\left|\frac{u}{a}\right| + c \quad ; \quad \forall u^2 > a^2$

Ex 3: Evaluate the following integrals:

1) $\int \frac{x^2}{\sqrt{1-x^6}}dx$
2) $\int \frac{dx}{\sqrt{9-x^2}}$
3) $\int \frac{x}{1+x^4}dx$
4) $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$
5) $\int \frac{dx}{x\sqrt{4x^2-1}}$
6) $\int \frac{2dx}{\sqrt{x}(1+x)}$
7) $\int \frac{dx}{1+3x^2}$
8) $\int \frac{2\cos x\,dx}{1+(\sin x)^2}$
9) $\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx$
10) $\int \frac{\tan^{-1}x}{1+x^2}\,dx$

Sol:

1) $\frac{1}{3}\int \frac{1}{\sqrt{1-(x^3)^2}}(3x^2\,dx) = \frac{1}{3}\sin^{-1}x^3 + c$
2) $\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3} + c$
3) $\frac{1}{2}\int \frac{2x}{1+(x^2)^2}dx = \frac{1}{2}\tan^{-1}x^2 + c$
4) $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx = \sin^{-1}(\tan x) + c$
5) $\int \frac{2dx}{2x\sqrt{4x^2-1}} = \sec^{-1}(2x) + c$
6) $\int \frac{2}{\sqrt{x}(1+x)}dx = 4\int \frac{\frac{1}{2}\sqrt{x}\,dx}{1+(\sqrt{x})^2} = 4\tan^{-1}\sqrt{x} + c$
7) $\frac{1}{\sqrt{3}}\int \frac{\sqrt{3}\,dx}{1+(\sqrt{3}x)^2} = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\,x) + c$
8) $2\int \frac{\cos x\,dx}{1+(\sin x)^2} = 2\tan^{-1}(\sin x) + c$

$u = x^3 \rightarrow du = 3x^2$
$u = x \rightarrow du = dx$
$u = x^2 \rightarrow du = 2x$
$u = \tan x \rightarrow du = \sec^2 x$
$u = 2x \rightarrow du = 2$
$u = \sqrt{x} \rightarrow du = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
$u = \sqrt{3}x \rightarrow du = \sqrt{3}$
$u = \sin x \rightarrow du = \cos x\,dx$

Email : alaa.khalid.abdalreda@uomus.edu.iq

Integrals of hyperbolic functions:

1) $\int \sinh u \, du = \cosh u + c$
2) $\int \cosh u \ du = \sinh u + c$
3) $\int \tanh u \, du = \ln(\cosh u) + c$
4) $\int \coth u \, du = \ln(\sinh u) + c$
5) $\int \operatorname{sech}^2 u \, du = \tanh u + c$
6) $\int \operatorname{csch}^2 u \, du = \coth u + c$
7) $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + c$
8) $\int \operatorname{csch} u \coth u \ du = -\operatorname{csch} u + c$

Ex 4: Evaluate the following integrals:

1) $\int \frac{\cosh(\ln x)}{x} dx$
2) $\int \sinh(2x+1) \ dx$
3) $\int \frac{\sinh x}{\cosh^4 x} dx$
4) $\int x \cosh(3x^2) \, dx$
5) $\int \sinh^4 x \cosh x \ dx$
6) $\int \operatorname{sech}^2(2x-3) \ dx$
7) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$
8) $\int (e^{ax} - e^{-ax}) \, dx$
9) $\int \frac{\sinh x}{1+\cosh x} \ dx$
10) $\int \operatorname{csch}^2 x \coth x \ dx$

Sol:

1) $\int \cosh(\ln x) \cdot \frac{dx}{x} = \sinh(\ln x) + c$
2) $\frac{1}{2}\int \sinh(2x+1) \, (2dx) = \frac{1}{2}\cosh(2x+1) + c$
3) $\int \frac{1}{\cosh^3 x} \frac{\sinh x}{\cosh x} \, dx = \int \operatorname{sech}^3 x \tanh x \ dx = -\int \operatorname{sech}^2 x \ (-\operatorname{sech} x \tanh x \, dx = -\frac{\operatorname{sech}^3 x}{3} + c$
4) $\frac{1}{6}\int \cosh(3x^2) \, (6x \, dx) = \frac{1}{6}\sinh(3x^2) + c$
5) $\int \sinh^4 x \, (\cosh x \ dx) = \frac{\sinh^5 x}{5} + c$

Email : alaa.khalid.abdalreda@uomus.edu.iq

Integrals of inverse hyperbolic functions:

1) $\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$

2) $\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$

3) $\int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + c & |u| < 1 \\ \coth^{-1} u + c & |u| > 1 \end{cases} = \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + c,$

4) $\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1} |u| + c$

5) $\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1} |u| + c$

Ex 5: *Evaluate the following integrals*:

1) $\int \frac{dx}{\sqrt{1+4x^2}}$ 2) $\int \frac{dx}{\sqrt{4+x^2}}$ 3) $\int \frac{dx}{1-x^2}$ 4) $\int \frac{dx}{x\sqrt{4+x^2}}$

5) $\int \frac{\sec^2 \theta}{\sqrt{\tanh^2 \theta - 1}} \, d\theta$

6) $\int \tanh^{-1}(\ln \sqrt{x}) \frac{dx}{x(1 - \ln^2 \sqrt{x}}$

Sol:

1) $\frac{1}{2} \int \frac{2\,dx}{\sqrt{1+4x^2}} = \frac{1}{2} \sinh^{-1} 2x + c$

2) $\int \frac{\frac{1}{2} dx}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = \sinh^{-1} \frac{x}{2} + c$

3) $\int \frac{dx}{1-x^2} = \tanh^{-1} x + c \quad if \quad |x| < 1$

4) $\int \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2} \int \frac{\frac{1}{2} dx}{\frac{x}{2}\sqrt{1+\left(\frac{x}{2}\right)^2}} = \frac{1}{2} \operatorname{csch}^{-1} \left| \frac{x}{2} \right| + c$

5) $\int \frac{1}{\sqrt{\tan^2 \theta - 1}} (\sec^2 \theta \; d\theta) = \cosh^{-1}(\tan \theta) + c$

Email : alaa.khalid.abdalreda@uomus.edu.iq

Email : alaa.khalid.abdalreda@uomus.edu.iq