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Signal and system
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2 Inverse Laplace Transforms

If $L(f(t)) = \phi(s)$, then $L^{-1}[\phi(s)] = f(t)$, where L^{-1} is called the inverse Laplace transform operator.

Some basic Inverse Laplace Formulas

1. $L^{-1}[\frac{1}{s}] = 1$
2. $L^{-1}[\frac{1}{s^n}] = \frac{t^{n-1}}{(n-1)!}$
3. $L^{-1}[\frac{1}{s-a}] = e^{at}$
4. $L^{-1}[\frac{1}{s+a}] = e^{-at}$
5. $L^{-1}[\frac{1}{s^2+a^2}] = \frac{\sin at}{a}$
6. $L^{-1}[\frac{s}{s^2+a^2}] = \cos at$
7. $L^{-1}[\frac{1}{s^2-a^2}] = \frac{\sinh at}{a}$
8. $L^{-1}[\frac{s}{s^2-a^2}] = \cosh at$
9. $L^{-1}[\phi(s-a)] = e^{at}L^{-1}[\phi(s)]$ (Shifting property)
10. $L^{-1}[-\frac{d}{ds}L(f(t))] = tf(t)$
11. $L^{-1}[\int_0^\infty L(f(t))\,ds] = \frac{f(t)}{t}$

2.1 Inverse Transformation Using Partial Fraction

Some times a rational function of 's' can be expressed as sum of simple rational functions using partial fractions and then inverse transformed using shifting property.

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
5.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
6.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where x^2+bx+c cannot be factorised further	$\frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$

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1. Find the inverse Laplace transform of the following

(a) $\frac{1}{s^5}$

(b) $\frac{3s+2}{s^2+9}$

(c) $\frac{5}{s^2+3s+7}$

(d) $\frac{3s+2}{(s-1)(s^2+1)}$

(e) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$

Ans.

(a)

$$L^{-1}[\frac{1}{s^5}] = \frac{t^{5-1}}{(5-1)!} = \frac{t^4}{24}$$

(b)

$$\begin{aligned} L^{-1}\left[\frac{3s+2}{s^2+9}\right] &= 3L^{-1}\left[\frac{s}{s^2+9}\right] + \frac{2}{3}L^{-1}\left[\frac{3}{s^2+9}\right] \\ &= 3\cos 3t + \frac{2}{3}\sin 3t \end{aligned}$$

(c)

$$\begin{aligned}
L^{-1}\left[\frac{5}{s^2+3s+7}\right] &= L^{-1}\left[\frac{5}{s^2+3s+7+(\frac{3}{2})^2-(\frac{3}{2})^2}\right] \\
&= L^{-1}\left[\frac{5}{(s+\frac{3}{2})^2+\frac{19}{2}}\right] \\
&= \frac{5}{\sqrt{\frac{19}{2}}}L^{-1}\left[\frac{\sqrt{\frac{19}{2}}}{(s+\frac{3}{2})^2+(\sqrt{\frac{19}{2}})^2}\right] \\
&= \frac{5\sqrt{2}}{\sqrt{19}}e^{-\frac{3t}{2}}L^{-1}\left(\frac{\sqrt{\frac{19}{2}}}{(s^2+\sqrt{\frac{19}{2}})^2}\right) \\
&= \frac{5\sqrt{2}}{\sqrt{19}}e^{-\frac{3t}{2}}\sin(\sqrt{\frac{19}{2}}t)
\end{aligned}$$

(d)

We have $\frac{3s+2}{(s-1)(s^2+1)} = \frac{A}{s-1}+\frac{Bs+C}{s^2+1}$

$\therefore 3s+2 = (s^2+1)A+(Bs+C)(s-1)$

Put s $=1$ then, $5 = A(2) \implies A = \frac{5}{2}$

equating the term containing s^2 then $0 = A+B \implies B = -A = -\frac{5}{2}$

Put s $=0$ then $2 = A+C \implies C = 2-A = \frac{-1}{2}$

$$\begin{aligned}
L^{-1}\left(\frac{3s+2}{(s-1)(s^2+1)}\right) &= L^{-1}\left(\frac{\frac{5}{2}}{s-1}+\frac{\frac{-5}{2}s+\frac{1}{2}}{s^2+1}\right) \\
&= \frac{5}{2}L^{-1}\left(\frac{1}{s-1}\right)-\frac{5}{2}L^{-1}\left(\frac{s}{s^2+1}\right)+\frac{1}{2}L^{-1}\left(\frac{1}{s^2+1}\right) \\
&= \frac{5}{2}e^t-\frac{5}{2}\cos t+\frac{1}{2}\sin t
\end{aligned}$$

(e)

$$\text{we have } \frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$ie,\ 2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

$$\begin{aligned} \text{Put s} = 1,\ 1 = A(2) &\implies A = \frac{1}{2} \\ \text{Put s} = 2,\ 1 = B(-1) &\implies B = -1 \\ \text{Put s} = 3,\ 5 = C(2) &\implies C = \frac{5}{2} \end{aligned}$$

$$\begin{aligned} L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right] &= \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}\left(\frac{1}{s-3}\right) \\ &= \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t} \end{aligned}$$

2.2 Convolution

Given any two functions $f(t)$ and $g(t)$ defined for $t > 0$, their convolution is defined as the function $h(t)$, where $h(t) = \int_0^t f(u)g(t-u)\,du$ and is denoted by $f(t) * g(t)$

Laplace transform of convolution

If $h(t) = f(t) * g(t)$ then $L(h(t)) = L(f * g) = L(f(t))L(g(t))$

Note:

If $L^{-1}(\phi(s)) = f(t)$ and $L^{-1}(\psi(s)) = g(t)$ then $L^{-1}(\phi(s)\psi(s)) = f(t) * g(t)$

1. Use convolution theorem to find the inverse Laplace transform of $\frac{1}{(s-a)(s-b)}$.

$$\begin{aligned}
L^{-1}\left(\frac{1}{(s-a)(s-b)}\right) &= L^{-1}\left(\frac{1}{s-a}\right) * L^{-1}\left(\frac{1}{s-b}\right) \\
&= e^{at} * e^{bt} = \int_0^t e^{au} e^{b(t-u)}\,du \\
&= \int_0^t e^{(a-b)t} e^{bt}\,du = e^{bt}\left[\frac{e^{(a-b)u}}{a-b}\right]_0^t \\
&= \frac{e^{bt}}{a-b}\left[e^{(a-b)t} - 1\right] \\
&= \frac{e^{at} - e^{bt}}{a-b}
\end{aligned}$$

2. Use convolution theorem to find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$.

$$\begin{aligned}
L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] &= L^{-1}\left(\frac{s}{s^2+a^2}\right) * L^{-1}\left(\frac{s}{s^2+b^2}\right) \\
&= \cos at * \cos bt = \int_0^t \cos(au)\cos(b(t-u))\,du \\
&= \int_0^t \frac{1}{2}\left(\cos(bt + (a-b)u) + \cos((a+b)u - bt)\right)\,du \\
&= \frac{1}{2}\left[\frac{\sin(bt + (a-b)u)}{a-b} + \frac{\sin((a+b)u - bt)}{a+b}\right]_0^t \\
&= \frac{1}{2}\left(\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right) \\
&= \frac{1}{2}\left(\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right)
\end{aligned}$$

$$(s^2-3s+2)L(y)=\frac{4}{s}+2s-3$$
$$L(y)=\frac{2s^2-3s+4}{s(s^2-3s+2)}$$
$$L(y)=\frac{2s^2-3s+4}{s(s-1)(s-2)}$$
$$y=L^{-1}\left[\frac{2s^2-3s+4}{s(s-1)(s-2)}\right]----(1)$$

We have by partial fraction $\frac{2s^2-3s+4}{s(s-1)(s-2)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-2}$
$2s^2-3s+4=A(s-1)(s-2)+Bs(s-2)+Cs(s-1)$

$$\begin{aligned} \text{Put s}=0\ 4 &= A(2) \implies A=2 \\ \text{Put s}=1\ 3 &= B(-1) \implies B=-3 \\ \text{put s}=2\ 6 &= C(2) \implies C=3 \end{aligned}$$

$$\begin{aligned} (1) \text{ becomes } y &= L^{-1}\left[\frac{2}{s}-\frac{3}{s-1}+\frac{3}{s-2}\right] \\ &= 2-3e^t+3e^{2t} \end{aligned}$$

2. Using Laplace transform solve $y''+5y'+6y=e^{-2t}$ Given $y(0)=y'(0)=1$

Ans.

Given diff. equation is $y''+5y'+6y=e^{-2t}$
Applying Laplace transform $L[y'']+5L[y']+6L[y]=L[e^{-2t}$
$$s^2L[y]-sy(0)-y'(0)+5sL[y]-5y(0)+6L[y]=\frac{1}{s+2}$$
$$(s^2+5s+6)L(y)-s-1-5=\frac{1}{s+2}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{2a\sin at - 2b\sin bt}{(a-b)(a+b)}\right)\\
&= \frac{a\sin at - b\sin bt}{a^2-b^2}
\end{aligned}
$$

3. Use convolution theorem to find the inverse Laplace transform of $\frac{1}{(s+1)(s-1)^2}$.

$$
\begin{aligned}
L^{-1}\left[\frac{1}{(s+1)(s-1)^2}\right] &= L^{-1}\left(\frac{1}{s+1}\right)*\left(\frac{1}{(s-1)^2}\right)\\
&= e^{-t}*te^{t} = te^{t}*e^{-t}\\
&= \int_0^t ue^{u}e^{-(t-u)}\,du\\
&= e^{-t}\int_0^t ue^{2u}\,du\\
&= e^{-t}\left[(u)(\frac{e^{2u}}{2}-\frac{e^{2u}}{4})\right]_0^t\\
&= e^{-t}\left(\frac{te^{2t}}{2}-\frac{e^{2t}}{4}+\frac{1}{4}\right)\\
&= \frac{te^{t}}{2}-\frac{e^{t}}{4}+\frac{e^{-t}}{4}
\end{aligned}
$$

2.3 Applications

Solution of Ordinary Differential Equations

Laplace transform converts an ordinary differential equation in the dependent variable y with a set of initial conditions into an algebraic equation in $L(y)$. Being an algebraic equation, the latter can be easily solved to get $L(y)$. Taking the inverse transform of $L(y)$, then yields the solution.

1. Using Laplace transform solve $y'' - 3y' + 2y = 4, \quad y(0) = 2, \quad y'(0) = 3$

Ans.

Given differential equation is $y'' - 3y' + 2y = 4$

Applying Laplace transform $L[y''] - 3L[y'] + 2L[y] = 4L[1]$

$$s^2L(y) - sy(0) - y'(0) - 3sL(y) - 3y(0) + 2L(y) = 4\frac{1}{s}$$

$$(s^2 - 3s + 2)L(y) - 2s - 3 + 6 = \frac{4}{s}$$

$$(s^2+5s+6)L(y) = \frac{1}{s+2} + s + 6$$
$$L(y) = \frac{s^2+8s+13}{(s+2)(s^2+5s+6)} = \frac{s^2+8s+13}{(s+2)^2(s+3)} ----(1)$$

We have by partial fraction $\frac{s^2+8s+13}{(s+2)^2(s+3)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+3}$

$s^2+8s+13 = A(s+2)(s+3) + B(s+3) + C(s+2)^2$

$$\begin{aligned} \text{Put s = -2 } 1 &= B(1) \implies B = 1 \\ \text{Put s = -3 } -2 &= C(1) \implies C = -2 \\ \text{Put s = 0 } 13 &= 6A + 3B + 4C \implies 6A = 13 - 3 + 8 \implies A = 3 \end{aligned}$$

$$\begin{aligned} \text{(1) becomes } y &= L^{-1}\left[\frac{3}{s+2} + \frac{1}{(s+2)^2} - \frac{2}{s+3}\right] \\ &= 3e^{-2t} + te^{-2t} - 2e^{-3t} \end{aligned}$$

3. Use Laplace transform solve $y'' + 2y' + 5y = e^{-t}\cos t$, given that $y(0) = 0$, $y'(0) = 1$

Given diff. equation is $y'' + 2y' + 5y = e^{-t}\cos t$

Applying Laplace transform $L(y'') + 2L(y') + 5L(y) = \frac{s+1}{(s+1)^2+1}$

$$s^2L(y) - sy(0) - y'(0) + 2sL(y) + 2y(0) + 5L(y) = \frac{s+1}{(s+1)^2+1}$$

$$(s^2+2s+5)L(y)-1=\frac{s+1}{(s+1)^2+1}$$

$$(s^2+2s+5)L(y)=\frac{s+1}{(s+1)^2+1}+1$$

$$L(y)=\frac{s+1}{((s+1)^2+1)(s^2+2s+5)}+\frac{1}{s^2+2s+5}$$

$$L(y)=\frac{s+1}{((s+1)^2+1)(s^2+2s+1+4)}+\frac{1}{(s^2+2s+1+4)}$$

$$y=L^{-1}\left[\frac{s+1}{((s+1)^2+1)((s+1)^2+2^2)}+\frac{1}{((s+1)^2+2^2)}\right]$$

$$y=e^{-t}L^{-1}\left[\frac{s}{(s^2+1)(s^2+2^2)}+\frac{1}{s^2+2^2}\right]$$

But we have $(s^2+2^2)-(s^2+1)=3$

$$y=e^{-t}\left[L^{-1}\left(\frac{s[(s^2+2^2)-(s^2+1)]}{s(s^2+1)(s^2+2^2)}\right)+L^{-1}\left(\frac{1}{s^2+2^2}\right)\right]$$

$$y=e^{-t}\left[\frac{1}{3}L^{-1}\left(\frac{s}{s^2+1}\right)-L^{-1}\left(\frac{s}{s^2+2^2}\right)\right]+e^{-t}\frac{\sin 2t}{2}$$

$$=\frac{e^{-t}}{3}\cos t-\frac{e^{-t}}{3}\cos 2t+\frac{e^{-t}}{2}\sin 2t$$

$$=e^{-t}\left(\frac{\cos t}{3}-\frac{\cos 2t}{3}+\frac{\sin 2t}{2}\right)$$

2.4 Exercise

1. Find the inverse Laplace transforms of the following

   (a) $\frac{3s-2}{s^2-5s+6}$
   (b) $\frac{2s-3}{s^2+6s+13}$
   (c) $\frac{6s+7}{(s+2)(s-1)^2}$
   (d) $\frac{2s+5}{(s+2)(s^2+9)}$
   (e) $\frac{s}{s^3+1}$

2. Use Convolution theorem find the inverse of the following

   (a) $\frac{s}{(s^2+1)^2}$
   (b) $\frac{1}{s(s+1)}$
   (c) $\frac{1}{(s^2+1)^2}$
   (d) $\frac{1}{s(s-3)^2}$
   (e) $\frac{s}{(s+2)(s^2+9)}$

3. Using Laplace transform solve following differential equations

   (a) $y'' - 2y' - 3y = \sin t$, given $y(0) = y'(0) = 0$
   (b) $y'' - 3y' + 2y = 4t + e^{3t}$, given that $y(0) = 1, \quad y'(0) = -1$
   (c) $y''' - 3y'' + 3y' - y = t^2e^t$, given $y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2$
   (d) $y'' - 3y' + 2y = 4$, given that $y(0) = 2, \quad y'(0) = 3$
   (e) $y'' + 2y' + 6y = 6te^{-t}$, given that $y(0) = 2 \quad y'(0) = 5$