1.5 Division by t

If $L(f(t)) = \phi(s)$, then $L(\frac{f(t)}{t}) = \int_s^\infty \phi\, ds$

1. Find the Laplace transforms of following

(a) $\frac{1-e^t}{t}$

(b) $\frac{\sin t}{t}$

(c) $\frac{e^{-t}\sin t}{t}$

Ans.

(a)

$$\begin{aligned} L(\frac{1-e^t}{t}) &= \int_s^\infty L(1-e^t)\, ds \\ &= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds \\ &= [\log s - \log(s-1)]_s^\infty \\ &= \left[\log\frac{s}{s-1}\right]_s^\infty = \left[\log\frac{1}{1-\frac{1}{s}}\right]_s^\infty \\ &= \log 1 - \log\frac{s}{s-1} = \log\frac{s-1}{s} \end{aligned}$$

(b)

$$\begin{aligned} L(\frac{\sin t}{t}) &= \int_s^\infty L(\sin t)\, ds \\ &= \int_s^\infty \frac{1}{s^2+1}\, ds = [\tan^{-1} s]_s^\infty \\ &= \tan^{-1}(\infty) - \tan^{-1}(s) = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s \end{aligned}$$

(c)

$$\begin{aligned} L(\frac{\sin t}{t}) &= \int_s^\infty L(\sin t\, ds \\ &= \int_s^\infty \frac{1}{(s)^2+1}\, ds \\ &= \cot^{-1}(s) \text{ from above problem} \\ \therefore L(\frac{e^{-t}\sin t}{t}) &= \cot^{-1}(s+1) \end{aligned}$$

1.6 Transforms of Derivatives

If $f'(t)$ is continuous and $L(f(t)) = \phi(s)$, then $L(f'(t)) = s\phi(s) - f(0)$ provided $\lim_{x\to\infty} e^{-st} f(t) = 0$

Note:

$$\begin{aligned} L(f'') &= s^2 L(f) - sf(0) - f'(0) \\ L(f''') &= s^3 L(f) - s^2 f(0) - sf'(0) - f''(0) \\ Ingeneral, L(f^{(n)}) &= s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - ..... - f^{(n-1)}(0) \end{aligned}$$

1.7 Transforms of Integrals

If $L(f(t)) = \phi(s)$, then $L(\int_0^t f(u)\,du) = \frac{\phi(s)}{s}$

1.8 Some Special Functions

Unit Step Function

The unit step function $u(t-a)$ is defined as $u(t-a) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t \geq a \end{cases}$, $a \geq 0$. The unit step function is also called the Heaviside function.

$$L(u(t-a)) = \frac{e^{-as}}{s}$$

Note:

Any piece wise continuous function $f(t) = \begin{cases} f_0(t) & 0 < t < t_1 \\ f_1(t) & t_1 < t < t_2 \\ f_2(t) & t_2 < t < t_3 \\ \cdot & \\ \cdot & \\ \cdot & \\ f_{(n-1)}(t) & t_{(n-1)} < t < t_n \\ f_n(t) & t_n < t < \infty. \end{cases}$ defined on $0 < t < \infty$ can be given by the single expression

$$f(t) = f_0(t)[u(t-0)-u(t-t_1)]+f_1(t)[u(t-t_1)-u(t-t_2)]+...+f_{(n-1)}(t)[u(t-t_{n-1})]+f_n(t)u(t-t_n).$$

1. Express the following function in terms of unit step function $f(t) = \begin{cases} 2+t^2 & if\, 0<t<2 \\ 6 & if\, 2<t<3 \\ \frac{2}{2t-5} & if\, t>3 \end{cases}$

   Ans.

$$\begin{aligned} f(t) &= (2+t^2)[u(t-0)-u(t-2)]+6[u(t-2)-u(t-3)]+\frac{2}{2t-5}\cdot u(t-3) \\ &= (2+t^2)u(t)+(4-t^2)u(t-2)+\left(\frac{32-12t}{2t-5}\right)\cdot u(t-3) \end{aligned}$$

Dirac delta function(Unit impulse function)

The unit impulse function denoted by $\delta(t)$ is defined by $\delta(t-a) = \begin{cases} 0, & t \neq 0 \\ \text{not defined} & t = 0. \end{cases}$

$$L(\delta(t-a)) = e^{-as}$$

1.9 Second shifting theorem

If f(t) has the Laplace transform $\phi(s)$ then $L(f(t-a)u(t-a)) = e^{-at}\phi(s)$

1. Find $L(\sin(t)u(t-\pi))$

$$\begin{aligned} \sin(t)u(t-\pi) &= \sin(t-\pi+\pi)u(t-\pi) = -\sin(t-\pi)u(t-\pi) \\ L(\sin(t)u(t-\pi)) &= -L(\sin(t-\pi)u(t-\pi)) \\ &= -e^{-\pi s}L(\sin t) = -\frac{e^{-\pi s}}{s^2+1} \end{aligned}$$

2. Find the Laplace Transform of $(t-1)^2u(t-1)$

   Ans.

$$\begin{aligned} L((t-1)^2u(t-1)) &= e^{-s}L(t^2) \\ &= e^{-s}\frac{2}{s^3} \end{aligned}$$

3. Express th following function in terms of unit step function and hence find its Laplace transform $f(t) = \begin{cases} t^2 & 1 < t \leq 2 \\ 4t & t > 2 \end{cases}$

Ans.

$$\begin{aligned}
f(t) &= t^2[u(t-1) - u(t-2)] + 4t[u(t-2)] \\
&= t^2u(t-1) - t^2u(t-2) + 4tu(t-2) \\
&= (t-1+1)^2u(t-1) - (t-2+2)^2u(t-2) + 4(t-2+2)u(t-2) \\
&= (t-1)^2u(t-1) + 2(t-1)u(t-1) + u(t-1) - (t-2)^2u(t-2) \\
&\quad -4(t-2)u(t-2) - 4u(t-2) + 4(t-2)u(t-2) + 8u(t-2) \\
&= (t-1)^2u(t-1) + 2(t-1)u(t-1) + u(t-1) \\
&\quad -(t-2)^2u(t-2) + 4u(t-2) \\
L(f(t)) &= e^{-s}L(t^2) + 2e^{-s}L(t) + e^{-s}L(1) - e^{-2s}L(t^2) + 4e^{-2s}L(t) \\
&= e^{-s}\frac{2}{s^3} + 2e^{-s}\frac{1}{s^2} + e^{-s}\frac{1}{s} - e^{-2s}\frac{2}{s^3} + 4e^{-2s}\frac{1}{s^2} \\
&= \frac{e^{-s}}{s^3}\left(s^2 + 2s + 2\right) + \frac{e^{-2s}}{s^3}\left(4s - 2\right)
\end{aligned}$$

1.10 Exercise

1. Find $L(e^{-t}t^2)$
2. Find $L(e^{2t}\cos 3t)$
3. Find $L(\sinh at \cos bt)$
4. Find $L(e^{2t}\sin^2 3t)$
5. Find $L(t\cos 2t)$
6. Find $L(e^{-t}t\cos 2t)$
7. Find $L(\frac{1-\cos t}{t})$
8. Find $L(tu(t-2)$
9. Find $L(e^{-2t}u(t-1)$
10. Express th following function in terms of unit step function and hence find its Laplace transform
$f(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$