



Functions and their inverse

A function f from a set D to a set Y is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in D$.

The set D of all possible input values is called the domain of the function. The set of all output values of $f(x)$ as x varies throughout D is called the range of the function.

A function $f(x)$ is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D .

The inverse function of a function f (also called the inverse of f) is a function that undoes the operation of f .

DEFINITION Suppose that f is a one-to-one function on a domain D with range R . The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D .

The symbol f^{-1} for the inverse of f is read “ f inverse.” The “ -1 ” in f^{-1} is *not* an exponent; $f^{-1}(x)$ does not mean $1/f(x)$. Notice that the domains and ranges of f and f^{-1} are interchanged.

The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x . This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y .
2. Interchange x and y , obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

EXAMPLE Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x .

Solution

1. *Solve for x in terms of y :* $y = \frac{1}{2}x + 1$ The graph is a straight line satisfying the horizontal line test (Fig. 1.60).

$$\begin{aligned} 2y &= x + 2 \\ x &= 2y - 2. \end{aligned}$$

2. *Interchange x and y :* $y = 2x - 2$.

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$.

To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

**-Inverse of trigonometric function**

$$Y = \sin^{-1} x \rightarrow x = \sin y$$

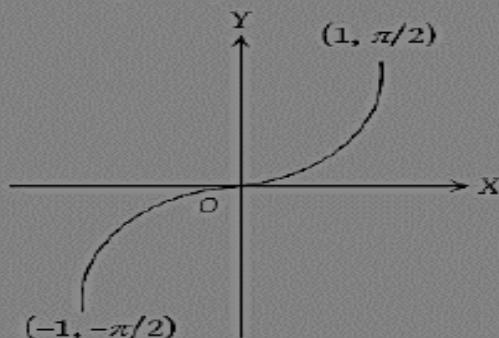
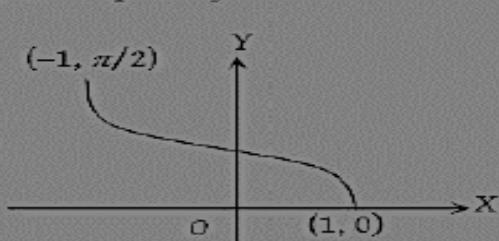
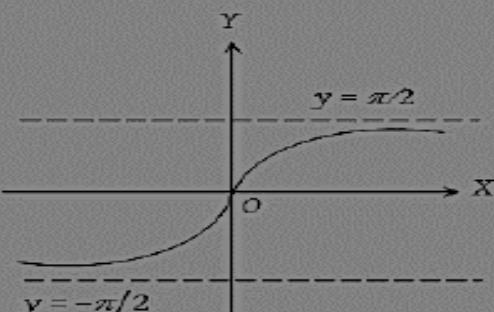
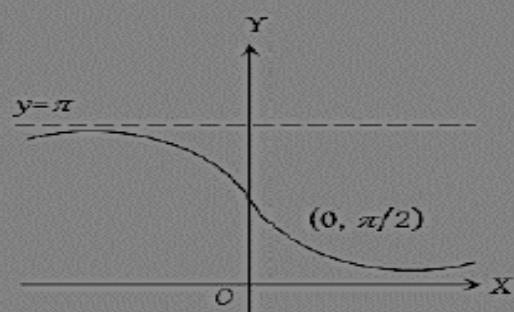
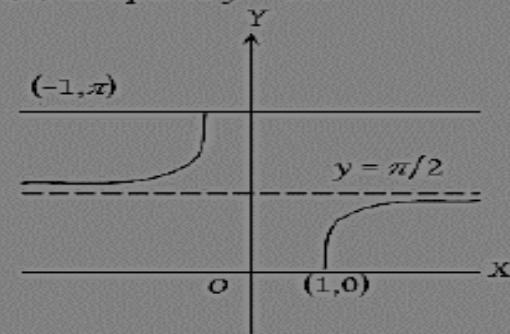
$$Y = \cos^{-1} x \rightarrow x = \cos y$$

$$Y = \tan^{-1} x \rightarrow x = \tan y$$

$$Y = \cot^{-1} x \rightarrow x = \cot y$$

$$Y = \sec^{-1} x \rightarrow x = \sec y$$

$$Y = \csc^{-1} x \rightarrow x = \csc y$$

(i) Graph of $y = \sin^{-1} x$ (ii) Graph of $y = \cos^{-1} x$ (iii) Graph of $y = \tan^{-1} x$ (iv) Graph of $y = \cot^{-1} x$ (v) Graph of $y = \sec^{-1} x$ (vi) Graph of $y = \csc^{-1} x$ 