



### Functions and their inverse

A function  $f$  from a set  $D$  to a set  $Y$  is a rule that assigns a unique (single) element  $f(x) \in Y$  to each element  $x \in D$ .

The set  $D$  of all possible input values is called the domain of the function. The set of all output values of  $f(x)$  as  $x$  varies throughout  $D$  is called the range of the function.

A function  $f(x)$  is one-to-one on a domain  $D$  if  $f(x_1) \neq f(x_2)$  whenever  $x_1 \neq x_2$  in  $D$ .

The inverse function of a function  $f$  (also called the inverse of  $f$ ) is a function that undoes the operation of  $f$ .

**DEFINITION** Suppose that  $f$  is a one-to-one function on a domain  $D$  with range  $R$ . The **inverse function**  $f^{-1}$  is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of  $f^{-1}$  is  $R$  and the range of  $f^{-1}$  is  $D$ .

The symbol  $f^{-1}$  for the inverse of  $f$  is read “ $f$  inverse.” The “ $-1$ ” in  $f^{-1}$  is *not* an exponent;  $f^{-1}(x)$  does not mean  $1/f(x)$ . Notice that the domains and ranges of  $f$  and  $f^{-1}$  are interchanged.

The process of passing from  $f$  to  $f^{-1}$  can be summarized as a two-step procedure.

1. Solve the equation  $y = f(x)$  for  $x$ . This gives a formula  $x = f^{-1}(y)$  where  $x$  is expressed as a function of  $y$ .
2. Interchange  $x$  and  $y$ , obtaining a formula  $y = f^{-1}(x)$  where  $f^{-1}$  is expressed in the conventional format with  $x$  as the independent variable and  $y$  as the dependent variable.

**EXAMPLE** Find the inverse of  $y = \frac{1}{2}x + 1$ , expressed as a function of  $x$ .

**Solution**

1. Solve for  $x$  in terms of  $y$ :  $y = \frac{1}{2}x + 1$  The graph is a straight line satisfying the horizontal line test (Fig. 1.60).

$$\begin{aligned} 2y &= x + 2 \\ x &= 2y - 2. \end{aligned}$$

2. Interchange  $x$  and  $y$ :  $y = 2x - 2$ .

The inverse of the function  $f(x) = (1/2)x + 1$  is the function  $f^{-1}(x) = 2x - 2$ .

To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$



**-Inverse of trigonometric function**

$$Y = \sin^{-1} x \rightarrow x = \sin y$$

$$Y = \cos^{-1} x \rightarrow x = \cos y$$

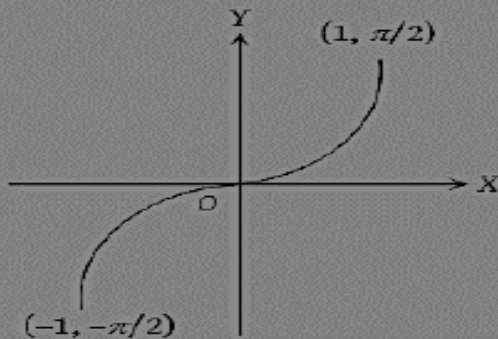
$$Y = \tan^{-1} x \rightarrow x = \tan y$$

$$Y = \cot^{-1} x \rightarrow x = \cot y$$

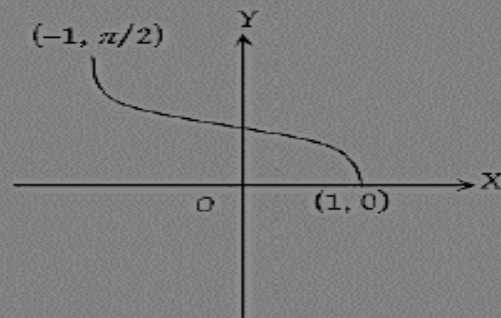
$$Y = \sec^{-1} x \rightarrow x = \sec y$$

$$Y = \csc^{-1} x \rightarrow x = \csc y$$

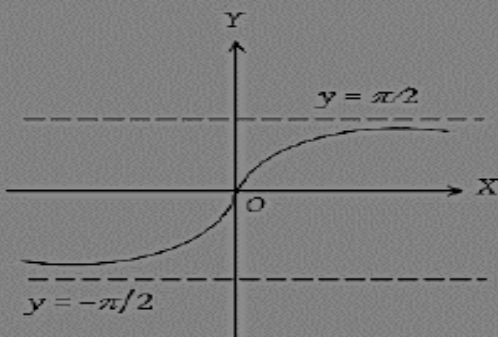
(i) Graph of  $y = \sin^{-1} x$



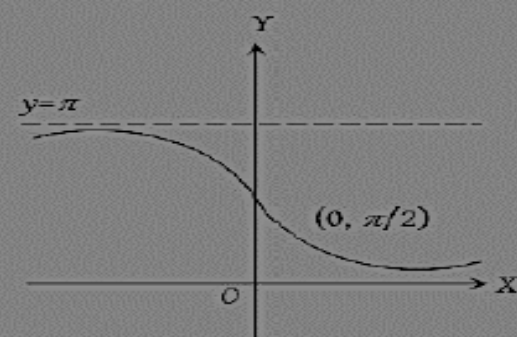
(ii) Graph of  $y = \cos^{-1} x$



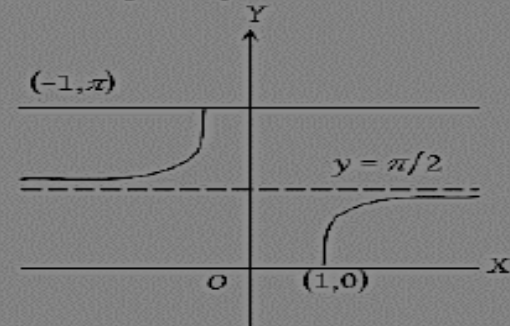
(iii) Graph of  $y = \tan^{-1} x$



(iv) Graph of  $y = \cot^{-1} x$



(v) Graph of  $y = \sec^{-1} x$



(vi) Graph of  $y = \csc^{-1} x$

