CONTENTS

LEARNING OBJECTIVES

By the end of this lecture, the student will be able to:

1. **Define electric current** and explain how it relates to the motion of electric charges.
2. **Apply the Biot–Savart Law** to determine the magnetic field intensity H produced by different current elements.
3. **Use Ampère's Circuital Law** to compute magnetic fields in symmetrical current distributions.
4. **Differentiate between electric and magnetic flux densities** and interpret their physical meaning.
5. **Evaluate magnetic flux** through different surfaces using integral expressions.
6. **Describe the conservation of magnetic flux** using Gauss's law for magnetostatics.

1 INTRODUCTION

Magnetostatics is the study of magnetic fields produced by steady (time-independent) electric currents. When charges move through a conductor at a constant rate, they generate a magnetic field in the surrounding space. These magnetic effects are essential in many engineering and technological applications, including electric machines, medical imaging systems, and communication devices.

The behavior of magnetic fields due to currents can be described using fundamental laws such as the BiotSavart Law and Ampère's Circuital Law. These laws allow engineers to compute the magnetic field intensity **H** produced by different current configurations.

Magnetic flux density **B**, another important quantity, represents how strongly magnetic field lines are distributed in space. The relationship between **B** and **H** involves the physical property known as permeability of the medium.

2 THE ELECTRIC CURRENT

Electric current is generally caused by the motion of electric charges:

The **current (in amperes)** through a given area is the electric charge passing through the area per unit time.

That is,

$$I = \frac{dQ}{dt} \tag{1}$$

Thus, in a current of one ampere, charge is being transferred at a rate of one coulomb per second. We now introduce the concept of current density J . If current ΔI flows through a planar surface ΔS, the current density is:

$$J = \frac{\Delta I}{\Delta S} \tag{2}$$

Or

$$\Delta I = J\Delta S \tag{3}$$

Assuming that the current density is perpendicular to the surface. If the current density is not normal to the surface,

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{S} \quad (4)$$

Thus, the total current flowing through a surface S is:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} \quad (5)$$

3 BIOT–SAVART'S LAW

Biot-Savart's law states that the differential magnetic field intensity dH produced at a point P, (as shown in Figure 1), by the differential current element Idl is proportional to the product $Id\mathbf{l}$ and the sine of the angle α between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element.

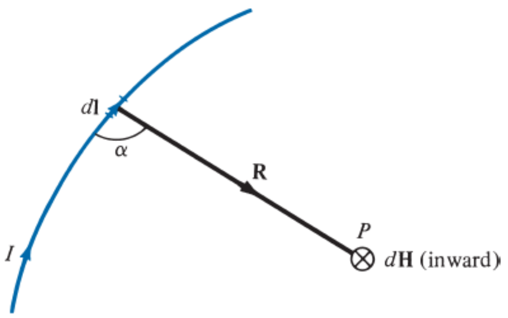


Figure 1: Magnetic field dH at P due to current element I dl.

That is,

$$dH \propto \frac{Idl\sin\,\alpha}{R^2} \quad (6)$$

Or

$$dH = \frac{kIdl\sin\,\alpha}{R^2} \tag{7}$$

Where k is the constant of proportionality. In SI units, $k = 1/4\pi$, so Equation (7) becomes:

$$dH = \frac{Idl\sin\,\alpha}{4\pi R^2} \tag{8}$$

It is easy to notice that $d\boldsymbol{H}$ is better put in vector form as:

$$d\boldsymbol{H} = \frac{Id\boldsymbol{l} \times \boldsymbol{a}_R}{4\pi R^2} = \frac{Id\boldsymbol{l} \times \boldsymbol{R}}{4\pi R^3} \tag{9}$$

The direction of d**H** can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current and the right-hand fingers encircling the wire in the direction of d**H** as shown in Figure 1.

we can have different current distributions: line current, surface current, and volume current as shown in Figure 2. If we define **K** as the surface current density in amperes per meter and **J** as the volume current density in amperes per meter squared, the source elements are related as

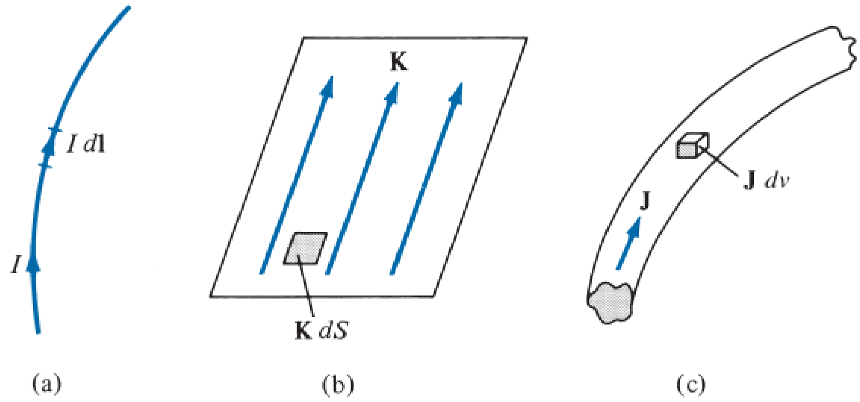


Figure 2: Current distributions: (a) line current, (b) surface current, (c) volume current.

Thus, in terms of the distributed current sources, the Biot-Savart's law as in Equation (9) becomes:

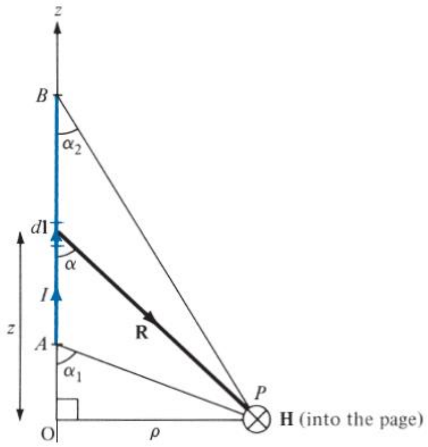
$$\mathbf{H} = \int_L \frac{I d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} \text{ (line current)}$$
$$\mathbf{H} = \int_S \frac{\mathbf{K} dS \times \mathbf{a}_R}{4\pi R^2} \text{ (surface current)} \qquad (10)$$
$$\mathbf{H} = \int_V \frac{\mathbf{J} dv \times \mathbf{a}_R}{4\pi R^2} \text{ (volume current)}$$

where $\mathbf{a}_R$ is a unit vector pointing from the differential element of current to the point of interest.

EXAMPLE 1 : Determine the field due to a straight current carrying filamentary conductor of finite length AB as in Figure below considering the following:

(a) point A is at $O(0,0,0)$ while B is at $(0,0,\infty)$.

(b) point A is at $(0,0,-\infty)$ while B is at $(0,0,\infty)$.

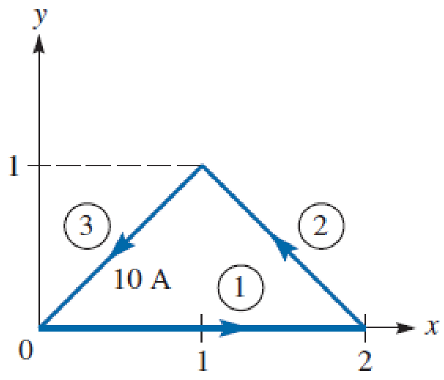


Solution

(a)

(b)

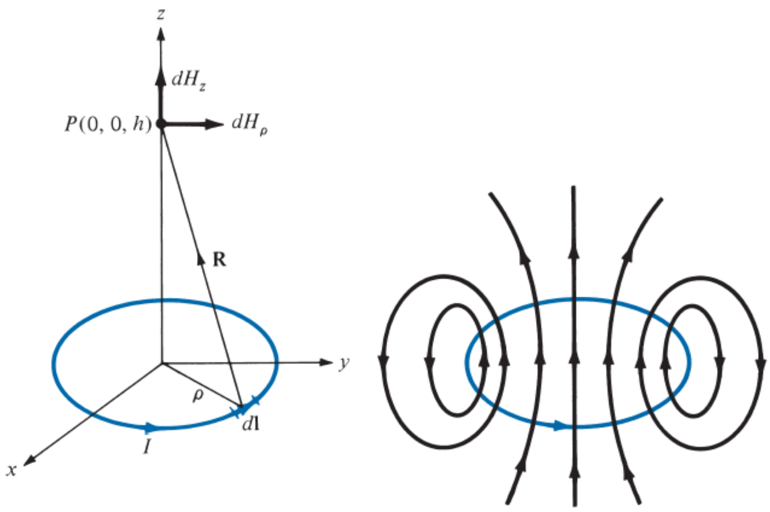
EXAMPLE 2 : The conducting triangular loop in Figure below carries a current of 10 A . Find H at (0,0,5) due to side 1 of the loop.



Solution

(a)

EXAMPLE 3 : A circular loop located on $x^2 + y^2 = 9, z = 0$ carries a direct current of 10 A along $\mathbf{a}_\phi$. Determine **H** at (0,0,4) and (0,0,−4).



(a) (b)

Solution

$$d\mathbf{H} = \frac{Id\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

where $d\mathbf{l} = \boldsymbol{\rho} d\boldsymbol{\phi} \mathbf{a}_{\boldsymbol{\phi}}, \mathbf{R} = (0,0,h) - (x,y,0) = -\boldsymbol{\rho}\mathbf{a}_{\boldsymbol{\rho}} + h\mathbf{a}_z$, and

$$d\mathbf{l} \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h d\phi \mathbf{a}_\rho + \rho^2 d\phi \mathbf{a}_z$$

Hence,

$$d\mathbf{H} = \frac{I}{4\pi[\rho^2 + h^2]^{3/2}}\left(\rho h d\phi \mathbf{a}_\rho + \rho^2 d\phi \mathbf{a}_z\right) = dH_\rho \mathbf{a}_\rho + dH_z \mathbf{a}_z$$

$$\mathbf{H} = \int dH_z \mathbf{a}_z = \int_0^{2\pi} \frac{I\rho^2 d\phi \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}} = \frac{I\rho^2 2\pi \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}}$$

or

$$\mathbf{H} = \frac{I\rho^2 \mathbf{a}_z}{2[\rho^2 + h^2]^{3/2}}$$

(a) Substituting $I = 10A, \rho = 3, h = 4$ gives

$$\mathbf{H}(0,0,4) = \frac{10(3)^2 \mathbf{a}_z}{2[9 + 16]^{3/2}} = 0.36\mathbf{a}_z \text{ A/m}$$

(b) Notice from $d\mathbf{l} \times \mathbf{R}$ in the Biot-Savart law that if h is replaced by $-h$, the z-component of $d\mathbf{H}$ remains the same while the ρ-component still adds up to zero due to the axial symmetry of the loop. Hence

$$\mathbf{H}(0,0,-4) = \mathbf{H}(0,0,4) = 0.36\mathbf{a}_z \text{ A/m}$$

4 AMPÈRE'S CIRCUIT LAW

Ampère's circuit law states that the line integral of **H** around a closed path is the same as the net current I_{enc} enclosed by the path.

In other words, the circulation of H equals I_{enc} ; that is,

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} \tag{11}$$

Ampère's law is similar to Gauss's law, since Ampère's law is easily applied to determine H when the current distribution is symmetrical. It should be noted that Equation (11) always holds regardless of whether the current distribution is symmetrical or not, but we can use the equation to determine H only when a symmetrical current distribution exists. Ampère's law is a special case of Biot-Savart's law; the former may be derived from the latter.

By applying Stokes's theorem to the left-hand side of Equation (11) we obtain

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} \qquad (12)$$

But,

$$I_{enc} = \int_S \boldsymbol{J} \cdot d\boldsymbol{S} \qquad (13)$$

Comparing the surface integrals in Equation (11) and (12) clearly reveals that:

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad (14)$$

5 APPLICATIONS OF AMPÈRE'S LAW

5.1 INFINITE LINE CURRENT

Consider an infinitely long filamentary current I along the z-axis as in Figure 3. To determine **H** at an observation, point P, we allow a closed path to pass through P. This path, on which Ampère's law is to be applied, is known as an *Amperian path.*

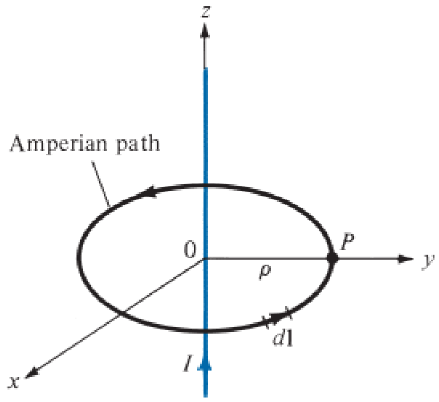


Figure 3: Ampère's law applied to an infinite filamentary line current.

$$I = \int_L H_\phi \mathbf{a}_\phi \cdot \rho d\phi \mathbf{a}_\phi = H_\phi \int_L \rho d\phi = H_\phi \cdot 2\pi\rho \qquad (15)$$

Or

$$\mathbf{H} = \frac{l}{2\pi\rho}\mathbf{a}_\phi \qquad (16)$$

5.2 INFINITE SHEET OF CURRENT

Consider an infinite current sheet in the $z = 0$ plane. If the sheet has a uniform current density $\mathbf{K} = K_y \mathbf{a}_y$ A/m as shown in Figure 4, applying Ampère's law to the rectangular closed path 1-2-3-4-1 (Amperian path) gives:

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} = K_y b \qquad (17)$$

The sheet can be regarded as consisting of such filamentary pairs so that the characteristics of **H** for a pair are the same for the infinite current sheet, that is,

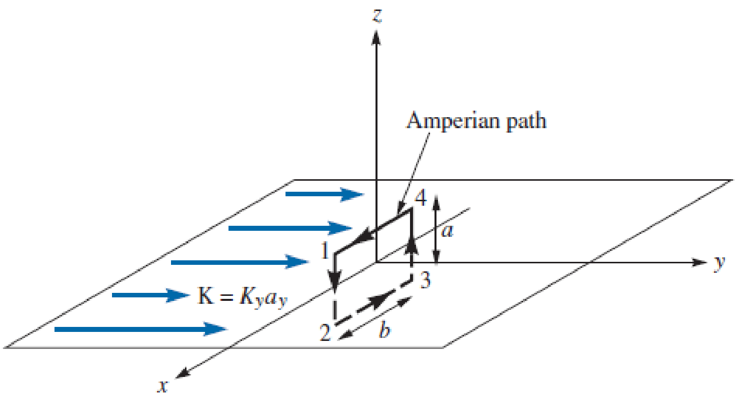


Figure 4: Application of Ampère's law to an infinite sheet.

$$\mathbf{H} = \begin{cases} H_o \mathbf{a}_x & z > 0 \\ -H_o \mathbf{a}_x & z < 0 \end{cases} \tag{18}$$

Where H_o is yet to be determined. Evaluating the line integral of H along the closed path gives:

$$\begin{aligned} \oint_L \mathbf{H} \cdot d\mathbf{l} &= \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \mathbf{H} \cdot d\mathbf{l} \\ &= 0(-a) + (-H_o)(-b) + 0(a) + H_o(b) \\ &= 2H_o b \end{aligned} \tag{19}$$

We obtain $H_o = \frac{1}{2} K_y$. Substituting H_o in Equation (18) gives:

$$\mathbf{H} = \begin{cases} \frac{1}{2} K_y \mathbf{a}_x, & z > 0 \\ -\frac{1}{2} K_y \mathbf{a}_x, & z < 0 \end{cases} \tag{20}$$

In general, for an infinite sheet of current density KA/m,

$$\mathbf{H} = \frac{1}{2} \mathbf{K} \times \mathbf{a}_n \tag{21}$$

Where $\mathbf{a}_n$ is a unit normal vector directed from the current sheet to the point of interest.

6 MAGNETIC FLUX DENSITY

The magnetic flux density B is similar to the electric flux density D. As $D = \varepsilon_0 E$ in free space, the magnetic flux density **B** is related to the magnetic field intensity **H** according to:

$$\mathbf{B} = \mu_o \mathbf{H} \tag{22}$$

Where μ_o is a constant known as the permeability of free space. The constant is in Henrys per meter (H/m) and $\mu_o = 4\pi \times 10^{-7}\,\mathrm{H/m}$.

The magnetic flux through a surface S is given by:

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} \tag{23}$$

Where the magnetic flux Ψ is in webers (Wb) and the magnetic flux density is in webers per square meter ($\mathrm{Wb/m^2}$) or teslas (T).

The total flux through a closed surface in a magnetic field must be zero; that is,

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{24}$$

This equation is referred to as the law of conservation of magnetic flux or Gauss's law for magnetostatic fields, just as $\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$ is Gauss's law for electrostatic fields. Although the magnetostatic field is not conservative, magnetic flux is conserved.

By applying the divergence theorem to Equation (24) we obtain:

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = \int_v \nabla\cdot\mathbf{B}dv = 0 \tag{25}$$

Or

$$\nabla\cdot\mathbf{B} = 0 \tag{26}$$

EXAMPLE 4 : In free space, $\mathbf{B} = \frac{20}{\rho}\sin^2\ \phi\mathbf{a}_z$ Wb/m^2. Determine the magnetic flux crossing the strip $z = 0, 1 < \rho < 2$ m$, 0 < \phi < \pi/4$.

Solution

$$\psi = \int_S \boldsymbol{B}\cdot d\boldsymbol{S} = \int_{\phi=0}^{\pi/4}\int_{\rho=1}^{2}\frac{20}{\rho}\sin^2\ \phi\rho d\rho d\phi = 20\int_1^2 d\rho\int_0^{\pi/4}\sin^2\ \phi d\phi$$

$$= 20(1)\int_0^{\pi/4}\frac{1}{2}(1-\cos\ 2\phi)d\phi = 10\left(\phi - \frac{1}{2}\sin\ 2\phi\right)\Big|_0^{\pi/4}$$

$$= 10\left(\frac{\pi}{4} - \frac{1}{2}\right) = \underline{\underline{2.854\ \text{Wb}}}$$

EXAMPLE 5 : If $\mathbf{B} = \frac{2}{r^3}\cos\ \theta\mathbf{a}_r + \frac{1}{r^3}\sin\ \theta\mathbf{a}_\theta$Wb/m^2, find the magnetic flux through the spherical cap $r = 1, \theta < \pi/3$.

Solution

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