Control Systems

- Time Response Analysis

Time Response Analysis

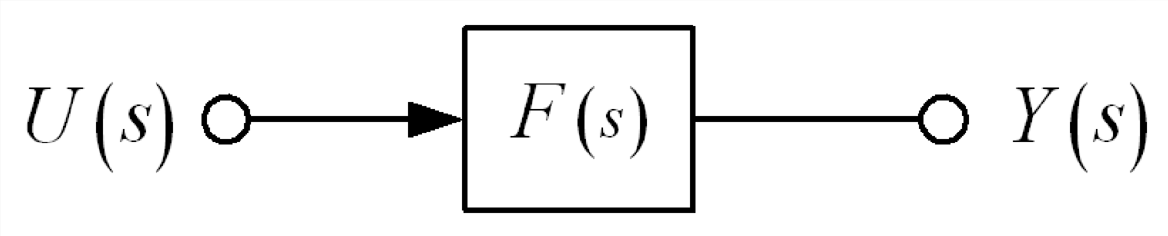
- 1- First step in analysing any control systems is to derive its mathematical model.
- 2- In analyzing and designing any control system we must have a basis of performance comparison with different control systems
- 3- This basis may be setup by specifying particular test input signals and by comparing the responses of various control systems to these input signals.
- 4- System is effected by changing the input test signal or its initial conditions.

Time Response Analysis

- 5- Typical test signals which commonly used in testing are of the type of: -Step functions - Ramp function - Impulse functions and Sinusoidal functions.
- 6- Time response analysis can be performed only for stable systems.
- 7- Time response of any system consists from Transient response and steady- state response.
- 8- Stability and steady state error are the most important characteristics in any control system.

Definition of Pole and Zero

- Consider the transfer function *F*(*s*):



$$\frac{Y(s)}{U(s)} = F(s) \quad \Rightarrow F(s) = \frac{B(s)}{A(s)}$$

Numerator polynomial
——————————
Denominator polynomial

- The system response is given by:

$$Y(s) = F(s)U(s) = \frac{B(s)}{A(s)}U(s)$$

- The **poles** are the values of *s* for which the denominator *A*(*s*) = 0.
- The **zeros** are the values of *s* for which the numerator *B*(*s*) = 0.

Effect of Pole Locations

- Consider the transfer function $F(s)$:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+\sigma} \qquad \Rightarrow Y(s) = \frac{1}{s+\sigma} U(s)$$

A form of first-order transfer function

- The impulse response will be an exponential function:

$$y(t) = e^{-\sigma t} \cdot 1(t)$$

  - When $\sigma>0$, the pole is located at $s < 0$,
    ➔ The exponential expression $y(t)$ decays.
    ➔ Impulse response is **stable**.
  - When $\sigma<0$, the pole is located at $s > 0$,
    ➔ The exponential expression $y(t)$ grows with time.
    ➔ Impulse response is referred to as **unstable**.

Effect of Pole Locations

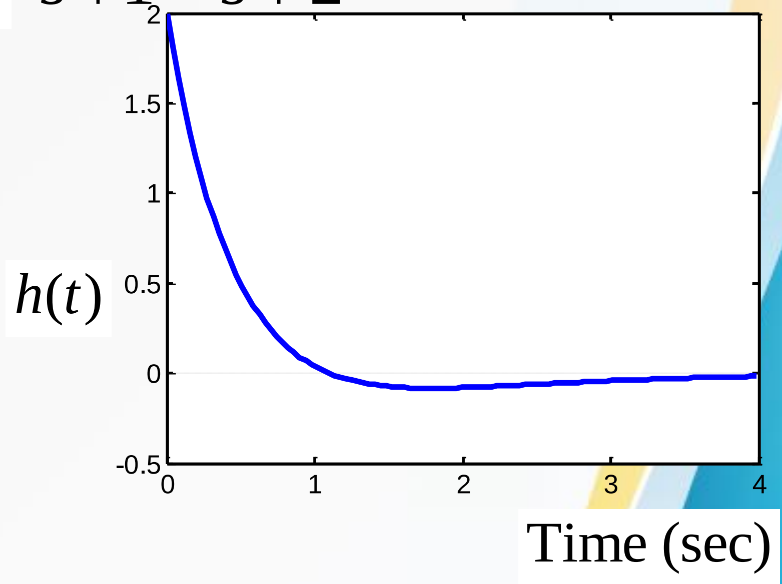
Example:

Find the impulse response of *H*(*s*),

$$H(s) = \frac{2s+1}{s^2+3s+2} = \frac{2s+1}{(s+1)(s+2)} = \frac{-1}{s+1} + \frac{3}{s+2}$$

$$h(t) = -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$h(t) = \underline{\underline{(-e^{-t} + 3e^{-2t})\cdot 1(t)}}$$

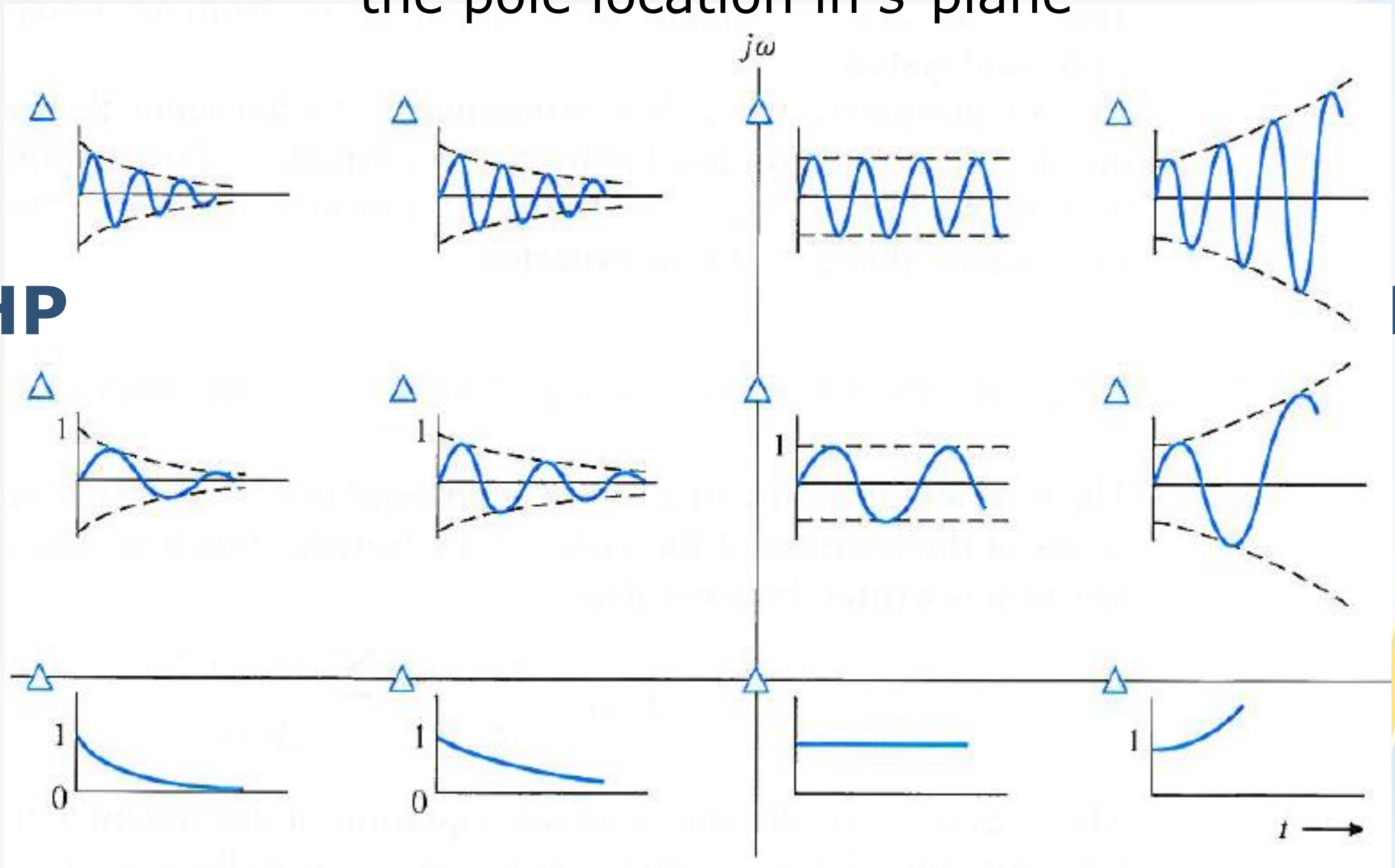


- **The terms e^{-t} and e^{-2t}, which are stable, are determined by the poles at $s = -1$ and -2. This is true for more complicated cases as well.**
- **In general, the response of a transfer function is determined by the locations of its poles.**

Effect of Pole Locations

Time function of impulse response assosiated with the pole location in *s*-plane

LHP

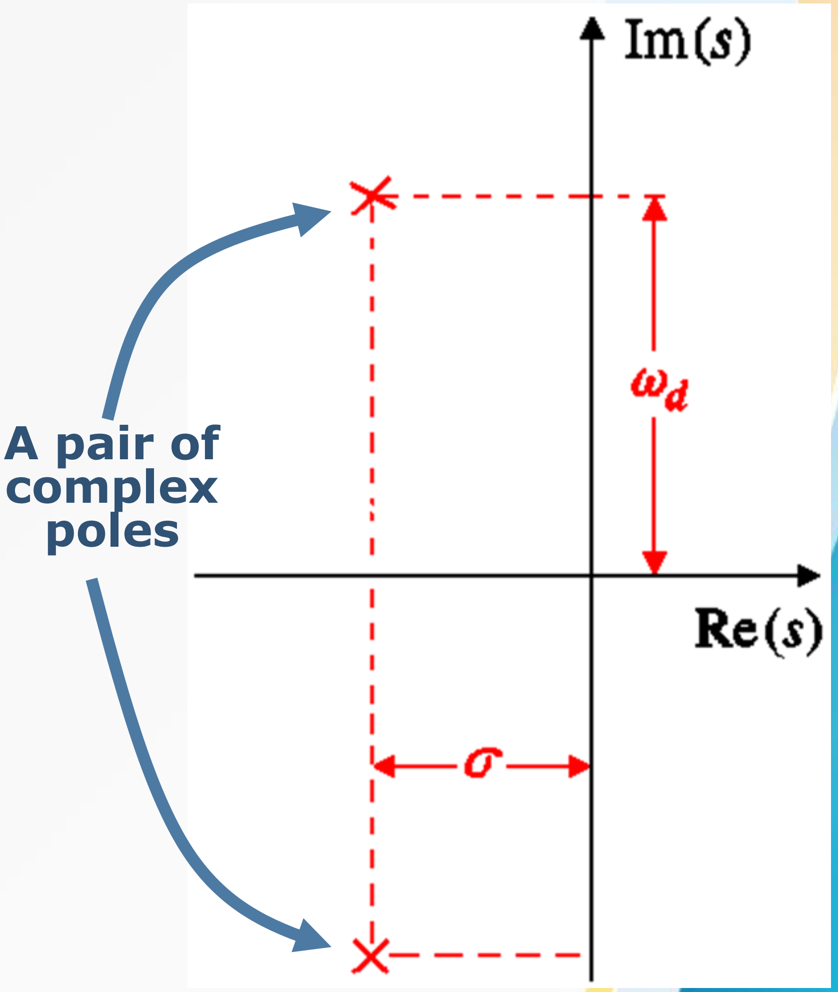


RHP

LHP : left half-plane
RHP : right half-plane

Representation of a Pole in *s*-Domain

- The position of a pole (or a zero) in *s*-domain is defined by its **real** and **imaginary parts,** Re(*s*) and Im(*s*).
- In rectangular coordinates, the complex poles are defined as $(-\sigma \pm j\omega_d)$.
- Complex poles always come in conjugate pairs.



Representation of a Pole in *s*-Domain

- The denominator corresponding to a complex pair will be:

$$A(s) = (s + \sigma - j\omega_d)(s + \sigma + j\omega_d)$$
$$= (s + \sigma)^2 + \omega_d^2$$

- On the other hand, the typical polynomial form of a second-order transfer function is:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

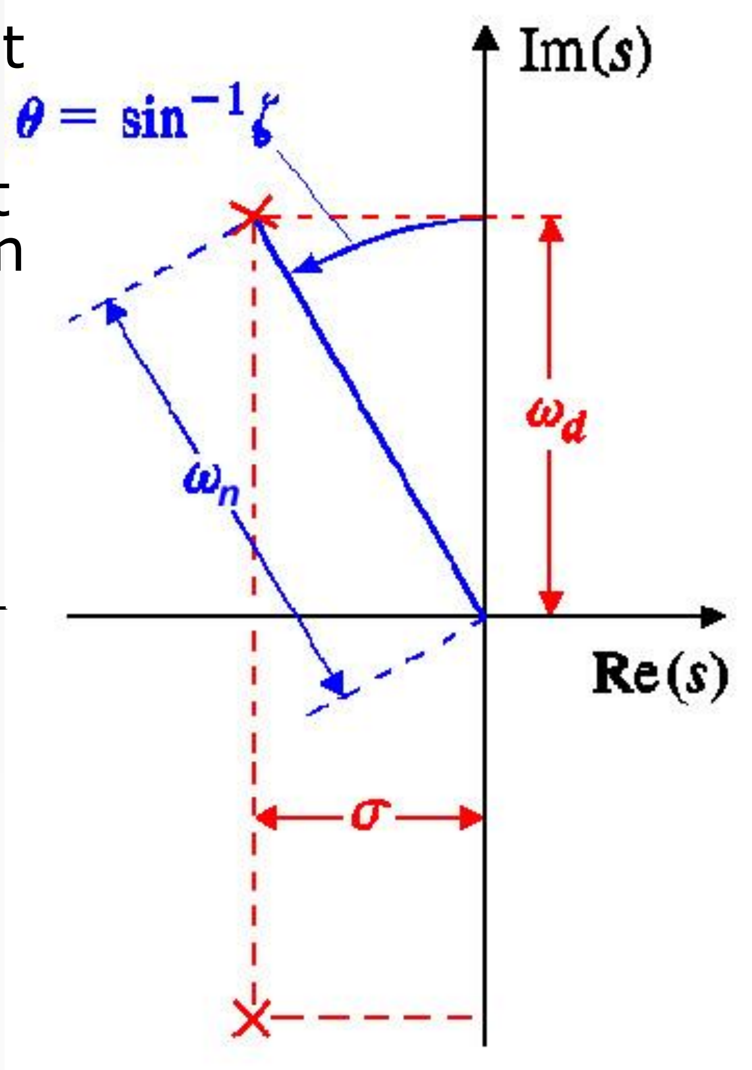
- Comparing *A*(*s*) and denominator of *H*(*s*), the correspondence between the parameters can be found:

$$\sigma = \zeta\omega_n \text{ and } \omega_d = \omega_n\sqrt{1-\zeta^2}$$

ζ : damping ratio
ω_n : undamped natural frequency
ω_d : damped frequency

Representation of a Pole in *s*-Domain

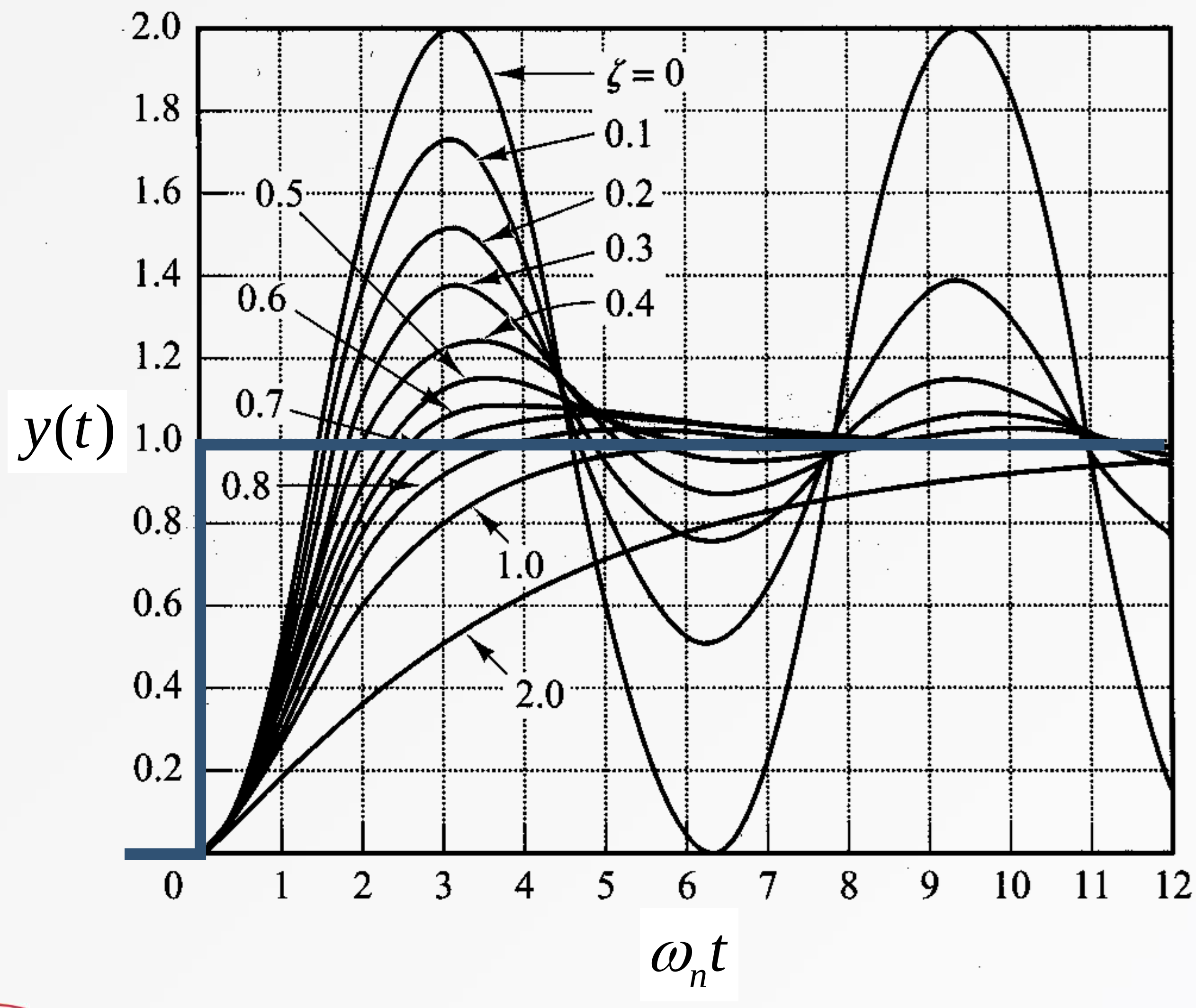
- Previously, in rectangular coordinates, the complex poles are at $(-\sigma \pm j\omega_d)$.
- In polar coordinates, the poles are at $(\omega_n, \sin^{-1}\zeta)$, as can be examined from the figure.



$$\sigma = \zeta\omega_n$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \quad \Leftrightarrow \quad \omega_n = \sqrt{\sigma^2 + \omega_d^2}$$

Unit Step Resonses of Second-Order System



Effect of Pole Locations

Example:

Find the correlation between the poles and the impulse response of the following system, and further find the exact impulse response.

$$H(s)=\frac{2s+1}{s^2+2s+5}$$

Since $H(s)=\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, $\omega_n^2=5 \Rightarrow \omega_n=\sqrt{5}=2.24 \text{ rad/sec}$

$$2\zeta\omega_n=2 \Rightarrow \zeta=0.447$$

The exact response can be otained from:

$$H(s)=\frac{2s+1}{s^2+2s+5}=\frac{2s+1}{(s+1)^2+2^2} \Rightarrow \text{poles at } s=-1\pm j2$$

Effect of Pole Locations

To find the inverse Laplace transform, the righthand side of the last equation is broken into two parts:

$$H(s) = \frac{2s+1}{(s+1)^2 + 2^2}$$

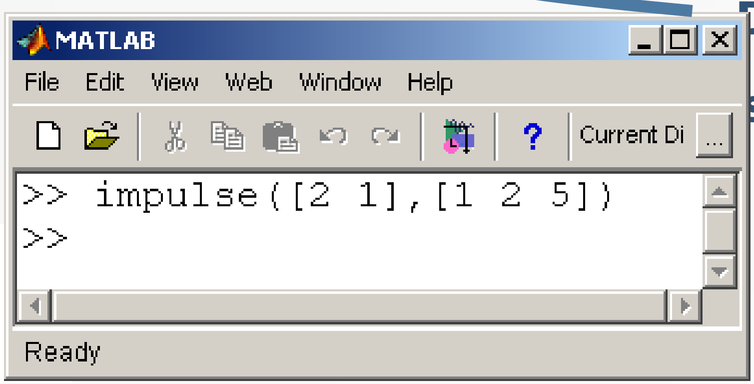
$$= 2\frac{s+1}{(s+1)^2+2^2} - \frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

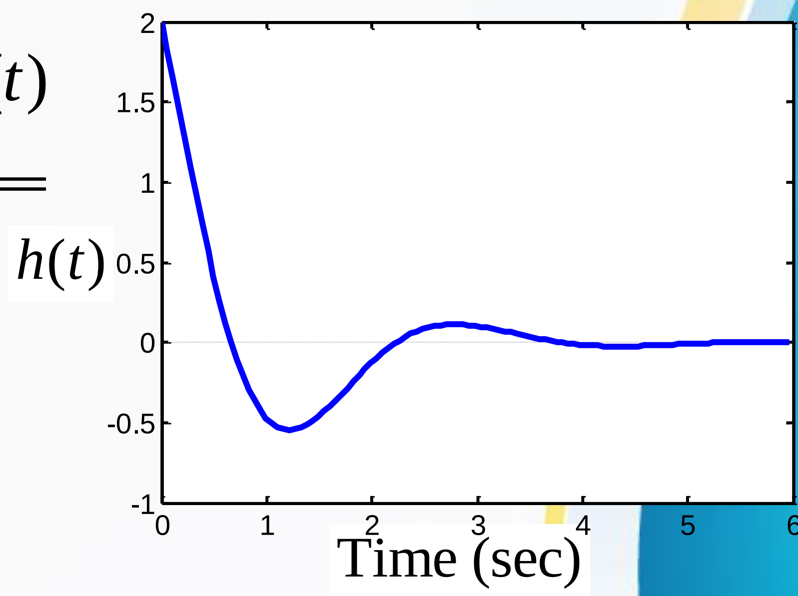
$$h(t) = \mathcal{L}^{-1}\left(H(s)\right)$$

$$= \left(2e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t\right)\cdot 1(t)$$

$f(t)$	$F(s)$
$\sin\omega t\ 1(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t\ 1(t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin\omega t\ 1(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t\ 1(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Damped
nusoidal
scillation



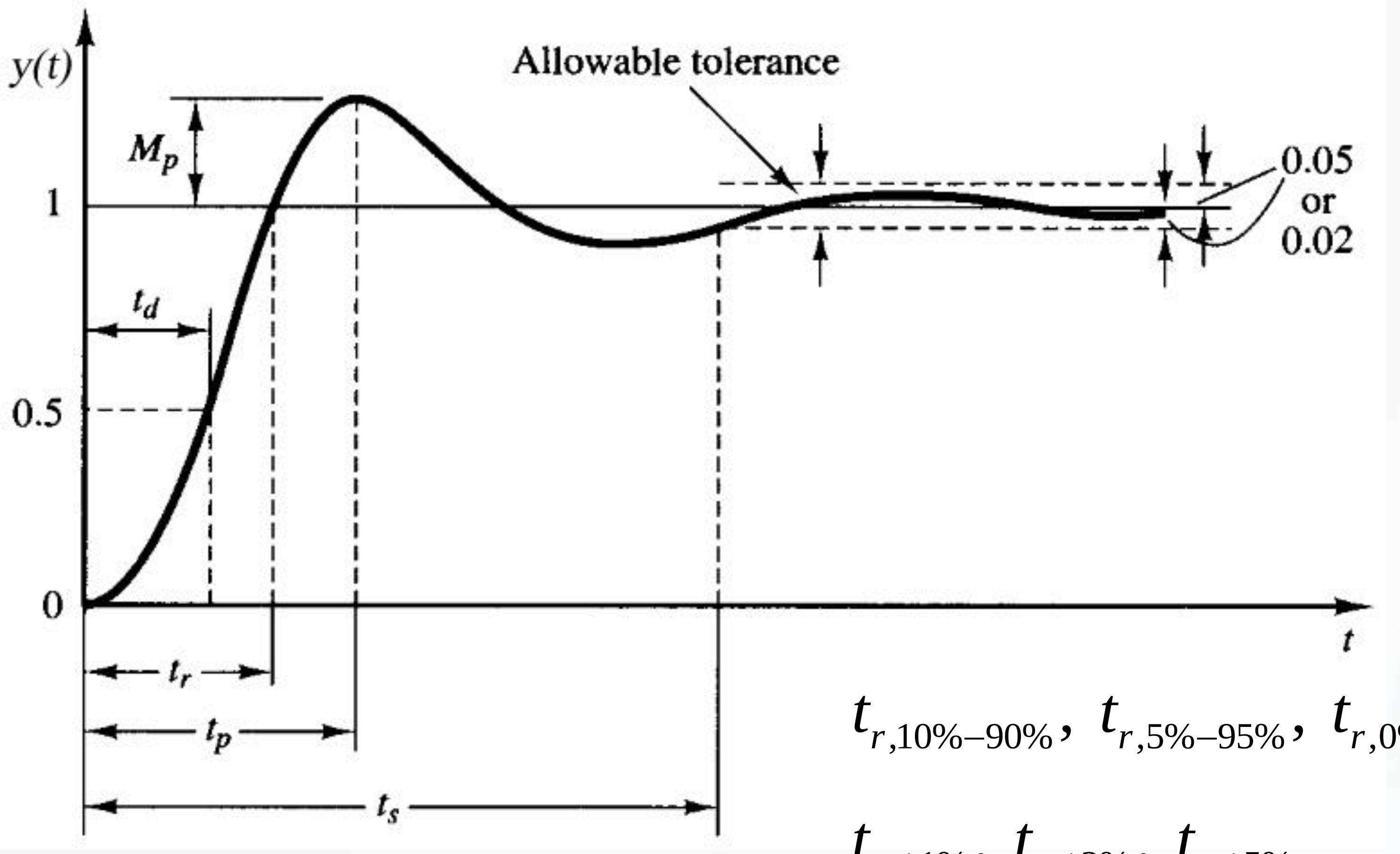


Time Domain Specifications

Specification for a control system design often involve certain requirements associated with the step response of the system:

1. **Delay time**, t_d, is the time required for the response to reach half the final value for the very first time.
2. **Rise time**, t_r, is the time needed by the system to reach the vicinity of its new set point.
3. **Settling time**, t_s, is the time required for the response curve to reach and stay within a range about the final value, of size specified by absolute percentage of the final value.
4. **Maximum Overshoot**, M_p, is the maximum peak value of the response measured from the final steady-state value of the response (often expressed as a percentage).
5. **Peak time**, t_p, is the time required for the response to reach the first peak of the overshoot.

Time Domain Specifications



$t_{r,10\%-90\%}$, $t_{r,5\%-95\%}$, $t_{r,0\%-100\%}$

$t_{s,\pm1\%}$, $t_{s,\pm2\%}$, $t_{s,\pm5\%}$

$$\%M_p = \frac{y(t_p) - y(\infty)}{y(\infty)} \cdot 100\%$$

First-Order System

- The step response of first-order system in typical form:

$$H(s)=\frac{1}{\tau s+1}$$

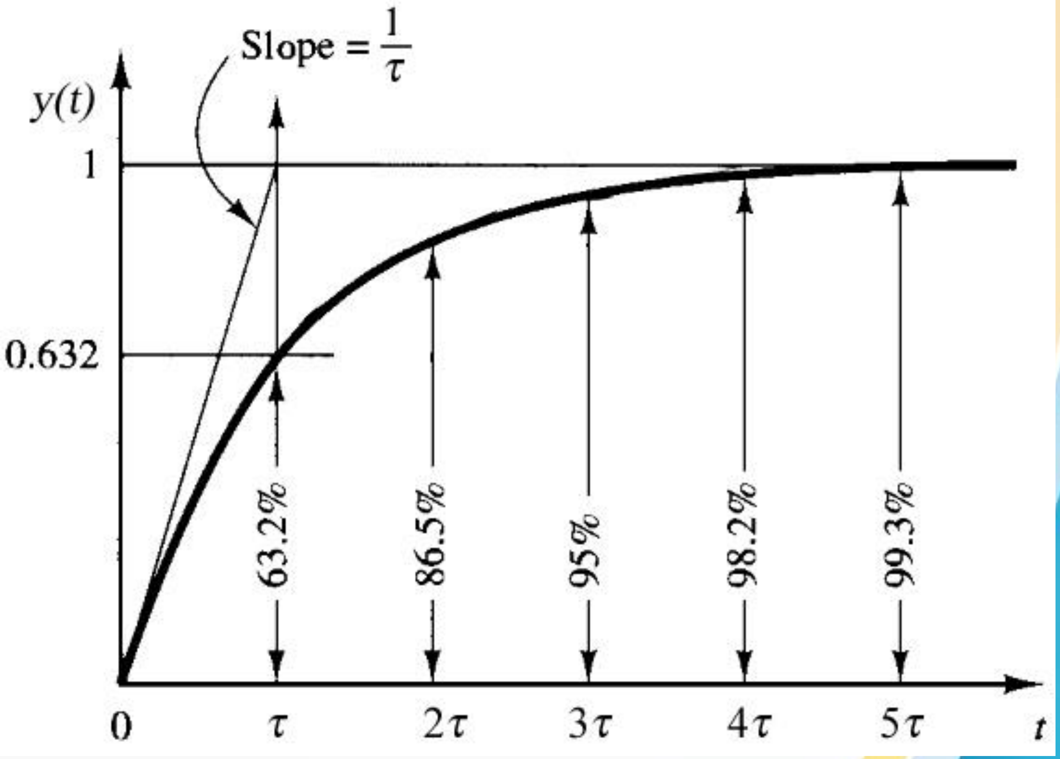
is given by:

$$Y(s)=\frac{1}{\tau s+1}\cdot\frac{1}{s}$$

$$=\frac{1}{s}-\frac{1}{s+(1/\tau)}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s+(1/\tau)}\right\}$$

$$y(t)=(1-e^{-t/\tau})\cdot 1(t)$$



- **τ : time constant**
- **For first order system, M_p and t_p do not apply**

Second-Order System

- The step response of second-order system in typical form:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

is given by:

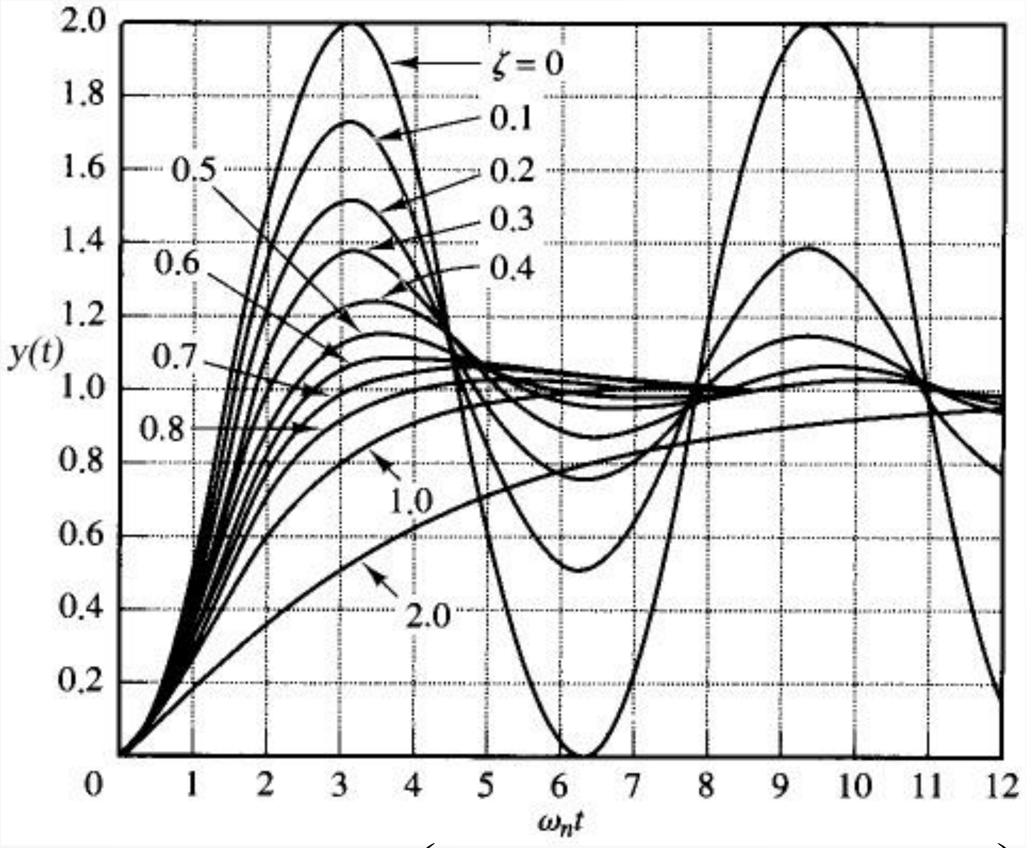
$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} \quad = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

Second-Order System



$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

- Time domain specification parameters apply for most second-order systems.
- Exception: overdamped systems, where $\zeta > 1$ (system response similar to first-order system).
- Desirable response of a second-order system is usually acquired with $0.4 < \zeta < 0.8$.

Rise Time

- The step response expression of the second order system is now used to calculate the rise time, $t_{r,0\%-100\%}$:

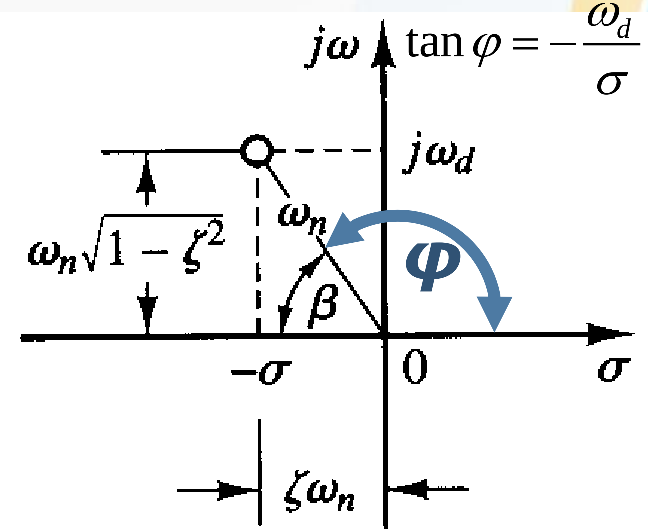
$$y(t_r) = 1 \equiv 1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right)$$

- Since $e^{-\zeta\omega_n t_r} \neq 0$, this condition will be fulfilled if:

$$\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0$$

or,

$$\tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$



$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_d}{\sigma}\right) = \frac{(\pi-\beta)}{\omega_d}$$

$$\sigma = \zeta\omega_n$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

t_r is a function of ω_d

Settling Time

- Using the following rule:

$$A\sin\alpha + B\cos\alpha = C\cos(\alpha-\beta),$$

with:

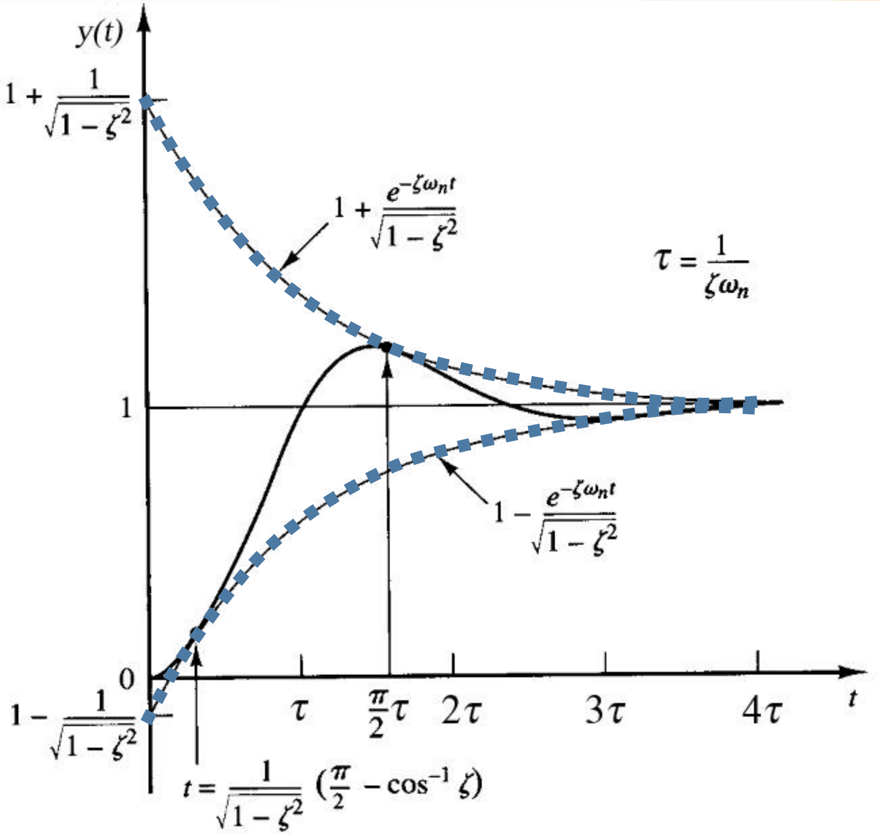
$$C=\sqrt{A^2+B^2},\ \beta=\tan^{-1}\left(\frac{A}{B}\right)$$

- The step response expression can be rewritten as:

$$y(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cdot\left(\cos(\omega_d t-\beta)\right)$$

where:

$$\beta=\tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$



***t_s* is a function of ζ**

$$\tau=\frac{1}{\zeta\omega_n}$$

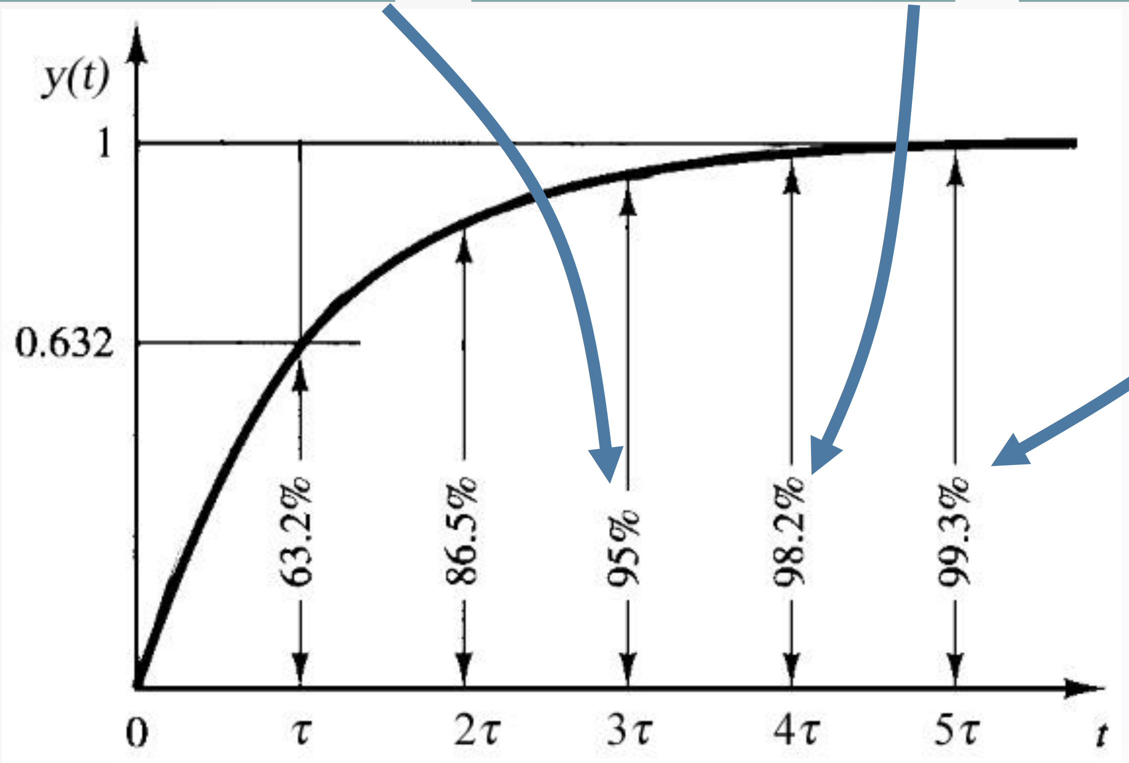
Settling Time

- The time constant of the envelope curves shown previously is $1/\zeta\omega_n$, so that the settling time corresponding to a certain tolerance band may be measured in term of this time constant.

$$t_{s,\pm5\%} = 3\tau = \frac{3}{\zeta\omega_n}$$

$$t_{s,\pm2\%} = 4\tau = \frac{4}{\zeta\omega_n}$$

$$t_{s,\pm1\%} = 5\tau = \frac{5}{\zeta\omega_n}$$



Peak Time

- When the step response $y(t)$ reaches its maximum value (maximum overshoot), its derivative will be zero:

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$y'(t) = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) +$$

$$e^{-\zeta\omega_n t}\left(\omega_d \sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)$$

$$y'(t) = e^{-\zeta\omega_n t}\left(\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \omega_d\right)\sin\omega_d t$$

Peak Time

- At the peak time,

$$y'(t_p) = e^{-\zeta\omega_n t_p}\left(\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \omega_d\right)\underbrace{\sin\omega_d t_p}_{\equiv 0} \equiv 0$$

$$\omega_d t_p = 0, \pi, 2\pi, 3\pi, \cdots$$

- Since the peak time corresponds to the first peak overshoot,

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

t_p is a function of ω_d

Maximum Overshoot

- Substituting the value of t_p into the expression for $y(t)$,

$$y(t_p) = 1 - e^{-\zeta\omega_n t_p}\left(\cos\omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_p\right)$$

$$y(t_p) = 1 - e^{-\zeta\omega_n \cdot \pi/\omega_d}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right) = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$M_p = y(t_p) - y(\infty)$$
$$= (1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}) - 1$$

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\%M_p = \frac{y(t_p) - y(\infty)}{y(\infty)} \cdot 100\%$$

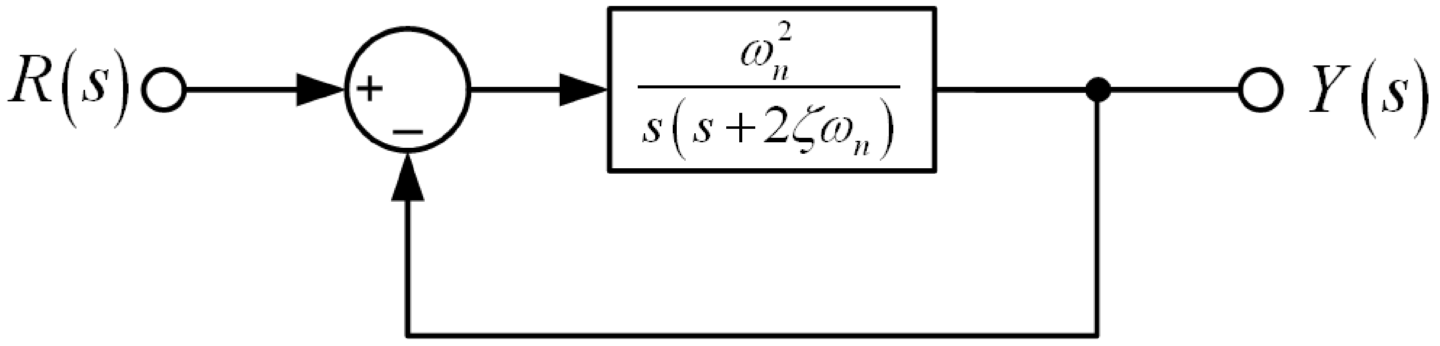
$$\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \cdot 100\%$$

if $y(\infty) = 1$

Example 1: Time Domain Specifications

Example:

Consider a system shown below with $\zeta = 0.6$ and $\omega_n = 5$ rad/s. Obtain the rise time, peak time, maximum overshoot, and settling time of the system when it is subjected to a unit step input.



After block diagram simplification,

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Standard form of second-order system

Example 1: Time Domain Specifications

$$\zeta = 0.6, \omega_n = 5 \text{ rad/s} \Rightarrow \omega_d = \sqrt{1-\zeta^2}\omega_n = \sqrt{1-0.6^2} \cdot 5 = 4 \text{ rad/s}$$

$$\Rightarrow \sigma = \zeta\omega_n = 0.6 \cdot 5 = 3 \text{ rad/s}$$

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_d}{\sigma}\right)$$ **In second quadrant**

$$= \frac{1}{4}\tan^{-1}\left(-\frac{4}{3}\right) = \frac{1}{4}(\pi - 0.927) = \underline{\underline{0.554 \text{ s}}}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = \underline{\underline{0.785 \text{ s}}}$$

Example 1: Time Domain Specifications

$$M_p = y(t_p) - y(\infty) = (1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}) - 1$$

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} = e^{-(0.6\cdot\pi)/0.8} = \underline{\underline{0.0948}}$$

$$\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \cdot 100\% = \underline{\underline{9.48\%}}$$

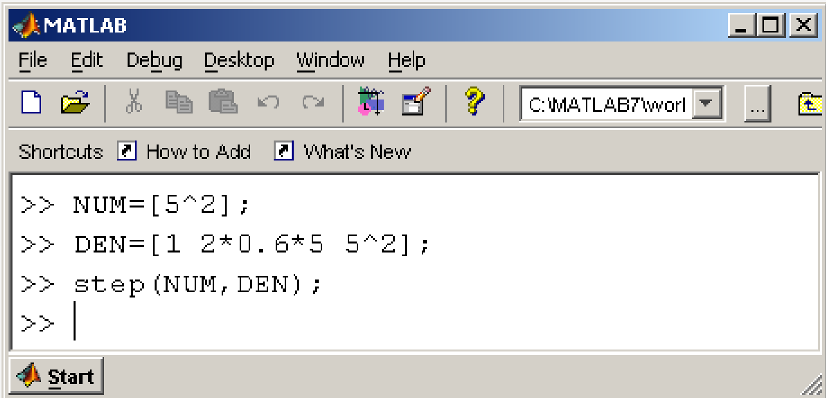
$$t_{s,\pm 2\%} = \frac{4}{\zeta\omega_n} = \frac{4}{0.6\cdot 5} = \underline{\underline{1.333\text{ s}}}$$

$$t_{s,\pm 5\%} = \frac{3}{\zeta\omega_n} = \frac{3}{0.6\cdot 5} = \underline{\underline{1\text{ s}}}$$

Check $y(\infty)$ for unit step input, if

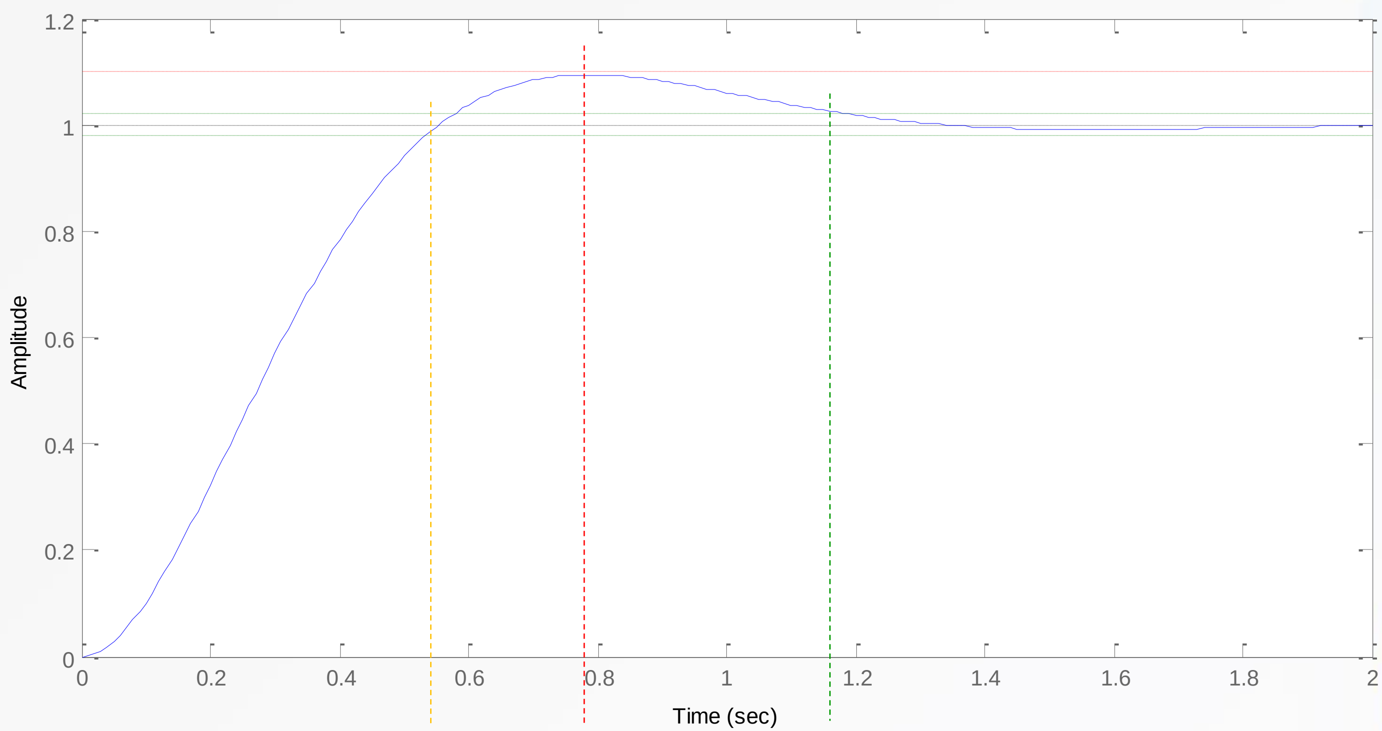
$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Example 1: Time Domain Specifications



$$t_r = 0.554 \text{ s}, \; t_p = 0.785 \text{ s}$$

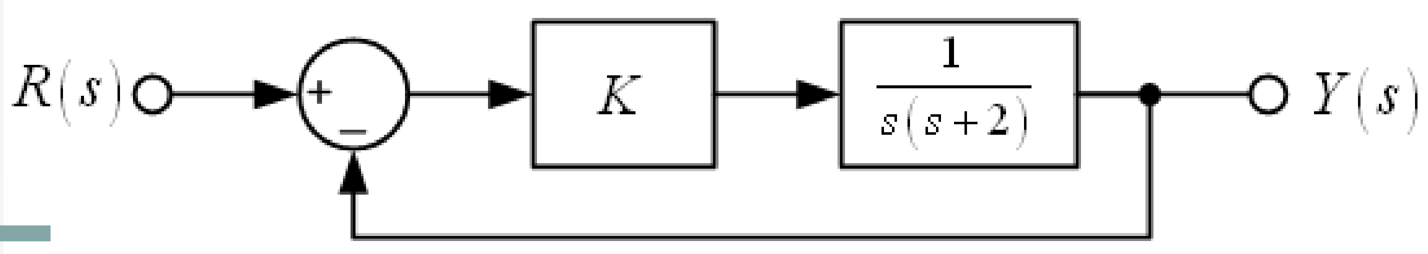
$$M_p = 9.48\%, t_s = 1.333 \text{ s}$$



Example 2: Time Domain Specifications

Example:

For the unity feedback system shown below, specify the gain *K* of the proportional controller so that the output *y*(*t*) has an overshoot of no more than 10% in response to a unit step.



$$\frac{Y(s)}{R(s)} = \frac{\frac{K}{s(s+2)}}{1+\frac{K}{s(s+2)}} = \frac{K}{s^2+2s+K}$$

$$\Rightarrow 2\zeta\omega_n = 2$$

$$\Rightarrow \omega_n^2 = K$$

$$\%M_p \le 10\% \Rightarrow e^{-\zeta\pi/\sqrt{1-\zeta^2}} \le 0.1 \quad \Rightarrow \zeta \ge 0.592$$

$$\Rightarrow \omega_n = \frac{1}{\zeta} \le \frac{1}{0.592} = 1.689$$

$$\Rightarrow K = \omega_n^2 \le 1.689^2 = 2.853$$

$$\therefore \underline{\underline{0 < K \le 2.853}}$$

Example 2: Time Domain Specifications